

Minimal Charts

Teruo NAGASE and Akiko SHIMA¹

Abstract

In this paper, we give definitions of three kinds of minimal charts, and we investigate properties of minimal charts and establish fundamental theorems characterizing minimal charts. To classify charts with two or three crossings we use the fundamental theorems. In the future paper, we give an enumeration of the charts with two crossings.

1 Introduction

Charts are oriented labeled graphs in a disk with three kinds of vertices called black vertices, crossings, and white vertices (see Section 2 for the precise definition of charts). Charts correspond to surface braids (see [6, Chapter 14] for the definition of surface braids). The closures of surface braids are embedded closed oriented surfaces in 4-space \mathbb{R}^4 (see [6, Chapter 23]). A *C-move* is a local modification between two charts in a disk. A C-move induces an ambient isotopy between the closures of the corresponding two surface braids. Two charts are said to be *C-move equivalent* if there exists a finite sequence of C-moves which modifies one of the two charts to the other.

We will work in the PL category or smooth category. All submanifolds are assumed to be locally flat. A *surface-link* is a closed surface embedded in \mathbb{R}^4 . A *2-link* is a surface-link each of whose connected component is a 2-sphere. A orientable surface-link is called a *ribbon surface-link* if it is the boundary of an immersed handlebodies with singularities which are mutually disjoint disks such that the preimage of each disk consists of a proper disk of the domain and a disk in the interior of the domain. In the words of charts, a ribbon surface-link is the closure of a surface braid which corresponds to a *ribbon chart*, a chart C-move equivalent to a chart without white vertices [4]. A chart is called a *2-link chart* if the chart represents the surface braid whose closure is a 2-link.

Kamada showed that any 3-chart is a ribbon chart [4]. Kamada's result was extended by Nagase and Hirota: Any 4-chart with at most one crossing is a ribbon chart [7]. We showed that any n -chart with at most one crossing is a ribbon chart [11]. We also showed that any 2-link chart with at most two crossings is a ribbon chart [12], [13].

Charts have strong conditions on orientations of arcs around vertices. In a small neighborhood of each white vertex, there are six short arcs, three consecutive arcs are oriented inward and the other three are outward. Among

¹ The second author is partially supported by Grant-in-Aid for Scientific Research (No.23540107), Ministry of Education, Science and Culture, Japan.

2010 Mathematics Subject Classification. Primary 57Q45; Secondary 57Q35.

Key Words and Phrases. 2-knot, chart, crossing.

six short arcs in a small neighborhood of a white vertex, a central arc of each three consecutive arcs oriented inward (resp. outward) is called a *middle arc* at the white vertex. Observing precisely middle arcs, orientations of edges, and a part of a chart cutting by a disk called a *tangle*, we shall prove the following theorem [14]: Any 2-link chart with at most three crossings is C-move equivalent to either a ribbon chart, or the disjoint union of free edges, hoops and a chart as shown in Fig. 1 or its “reflection”. In this paper we establish fundamental theorems characterizing *c*-minimal charts, *w*-minimal charts and *cw*-minimal charts. For the classification theorem above, we use the fundamental theorems.

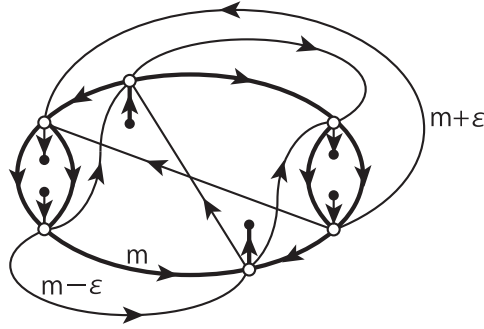


Figure 1: The letter m is a label and $\varepsilon = \pm 1$.

For a 4-chart as shown in Fig. 1 we obtain a 2-twist spun trefoil by setting $m = 2$. It is well known that the 2-knot is not a ribbon 2-knot. On the other hand, Hasegawa showed that if a non-ribbon chart representing a 2-knot is minimal, then the chart must possess at least six white vertices [2] where a minimal chart Γ means its complexity $(w(\Gamma), -f(\Gamma))$ is minimal among the charts C-move equivalent to the chart Γ with respect to the lexicographic order of pairs of integers, $w(\Gamma)$ is the number of white vertices in Γ , and $f(\Gamma)$ is the number of free edges in Γ . Nagase, Ochiai, and Shima showed that there does not exist a minimal chart with five white vertices [18]. Nagase and Shima show that there does not exist a minimal chart with seven white vertices [8],[9],[10],[15]. Ishida, Nagase, and Shima showed that any minimal chart with exactly four white vertices is C-move equivalent to a chart in two kinds of classes [3].

Let Γ be a chart. An edge of Γ is the closure of a connected component of the set obtained by taking out all of white vertices and crossings from Γ . For each label m , we denote by Γ_m the “subgraph” of Γ consisting of all the edges of label m and their vertices. We assume

an *edge* of Γ_m is the closure of a connected component of the set obtained by taking out all the white vertices from Γ_m .

Thus

any vertex of Γ_m is a black vertex or a white vertex. Hence any crossing

of Γ is not considered as a vertex of Γ_m . In this paper, an edge of label m means an edge of Γ .

Let Γ be a chart, and m a label of Γ . A simple closed curve in Γ_m is called a *cycle of label m* . Let e be an edge of Γ or an edge of Γ_m . Then the edge e is called a *loop* if it is a closed edge containing exactly one white vertex. The edge e is called a *terminal edge* if it contains a white vertex and a black vertex. The edge e is said to be *middle at a white vertex v* if it is not a loop and contains a middle arc at the vertex v .

Let Γ be a chart. Let C be a cycle of label m in Γ bounding a disk E . Let k be a label of Γ with $|m - k| \leq 1$. Then an edge e of Γ_k is called an *outside (resp. inside) edge for C* provided that

- (i) $e \cap C$ consists of one white vertex or two white vertices, and
- (ii) $e \not\subset E$ (resp. $e \subset E$).

Note that an outside (or inside) edge of Γ_m for C is not a loop.

For a cycle C of label m in a chart Γ , we define

$$\begin{aligned} \mathcal{O}(C) &= \{w \in C \mid w \text{ is a white vertex of an outside edge of } \Gamma_m \text{ for } C\}, \\ \mathcal{O}_M(C) &= \{w \in \mathcal{O}(C) \mid \text{there exists an outside edge of } \Gamma_m \text{ for } C \text{ middle at } w\}, \\ \mathcal{I}(C) &= \{w \in C \mid w \text{ is a white vertex of an inside edge of } \Gamma_m \text{ for } C\}, \\ \mathcal{I}_M(C) &= \{w \in \mathcal{I}(C) \mid \text{there exists an inside edge of } \Gamma_m \text{ for } C \text{ middle at } w\}. \end{aligned}$$

Let Γ be a chart, and E a disk bounded by a cycle of label m in Γ . Then the disk E is called a *three-color disk* provided that

- (i) the disk E does not contain any crossings, and
- (ii) $\Gamma \cap E \subset \Gamma_{m-1} \cup \Gamma_m \cup \Gamma_{m+1}$.

Let Γ be a chart. Let e_1 and e_2 be edges of Γ which connect two white vertices w_1 and w_2 where possibly $w_1 = w_2$. Suppose that the union $e_1 \cup e_2$ bounds an open disk E . Then $Cl(E)$ is called a *bigon* of Γ provided that any edge containing w_1 or w_2 does not intersect the open disk E (see Fig. 2). Note that neither e_1 nor e_2 contains a crossing.

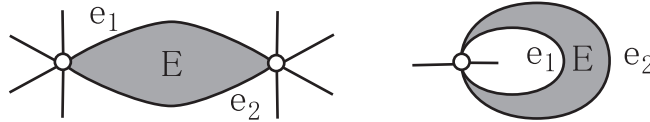


Figure 2:

Let Γ be a chart. Let $c(\Gamma)$ and $b(\Gamma)$ be the number of crossings, and the number of bigons of Γ respectively. The 4-tuple $(c(\Gamma), w(\Gamma), -f(\Gamma), -b(\Gamma))$ is called a *c-complexity* of the chart Γ . The 4-tuple $(w(\Gamma), -f(\Gamma), c(\Gamma), -b(\Gamma))$ is

called a *w-complexity* of the chart Γ . The 3-tuple $(c(\Gamma) + w(\Gamma), -f(\Gamma), -b(\Gamma))$ is called a *cw-complexity* of the chart Γ (see [4] for complexities of charts).

A chart Γ is said to be *c-minimal* (resp. *w-minimal* or *cw-minimal*) if its *c-complexity* (resp. *w-complexity* or *cw-complexity*) is minimal among the charts which are C-move equivalent to the chart Γ with respect to the lexicographical order of the 4-tuple of the integers. If a chart is *c-minimal*, *w-minimal* or *cw-minimal*, then we say that the chart is *minimal* in this paper.

Three-color disks often appear in charts. For example, let m be any label of a chart Γ , and C a cycle of label m in Γ bounding a disk E without crossings. If Γ is a minimal chart, then we can show that E is a three-color disk (see Section 12). This indicates that it is important to investigate cycles of label m bounding three-color disks.

For a set X , the number of all the elements in the set X is denoted by $|X|$.

Theorem 1.1 *Let Γ be a minimal chart. Let C be a cycle of label m in Γ bounding a three-color disk E . If $\Gamma_m \cap E$ is connected, then we have*

$$|\mathcal{I}_M(C)| + 2 \leq |\mathcal{O}_M(C)|.$$

Let Γ be a chart, and m a label of Γ . A simple arc P in Γ_m is called a *path* of label m provided that the endpoints of P are vertices of Γ . A simple arc P^* in Γ_m is called a *pseudo path* of label m provided that

- (i) P^* contains at least one vertex of Γ , and
- (ii) the endpoints of P^* are not black vertices, crossings, nor white vertices.

Let P^* be a pseudo path of label m in a chart Γ . A disk Δ is called a *side-disk* of P^* provided that $P^* \subset \partial\Delta$ (see Fig. 3). Let N be a regular neighborhood of P^* in the side-disk Δ . Let e be an edge of Γ , and γ the closure of a connected component of $e \cap \text{Int}N$. If γ contains a white vertex in P^* , then γ is called a *side-arc* of P^* with respect to Δ . A side-arc is said to be *at a vertex* v if it contains the vertex v . Similarly, a side-arc is said to be *middle* at a vertex v if it contains a middle arc at v .

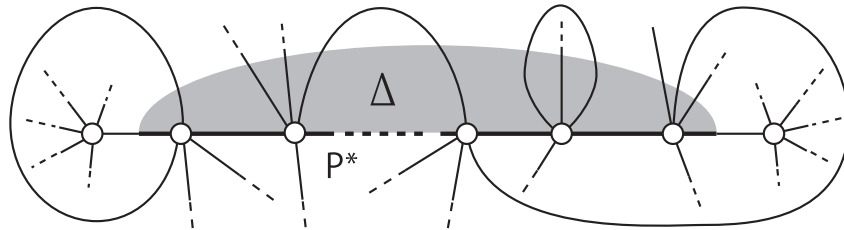


Figure 3: The thick line is a pseudo path P^* , and the gray area is a side-disk Δ .

Let P^* be a pseudo path of label m in a chart with a side-disk Δ . Then the pseudo path P^* is said to be *inward* (*resp. outward*) with respect to Δ provided that

- (i) P^* does not contain any crossings, and
- (ii) for each vertex v in P^* , any side-arc at v with respect to Δ is oriented inward (*resp. outward*) at v .

An inward pseudo path and an outward pseudo path are called *I/O pseudo paths*.

Let Γ be a chart, and m a label of Γ . An edge of Γ_m is called a *normal* edge if it contains two white vertices.

Let Γ be a chart, D a disk. The pair $(\Gamma \cap D, D)$ is called a *tangle* provided that

- (i) ∂D does not contain any white vertices, black vertices nor crossings of Γ ,
- (ii) if an edge of Γ intersects ∂D , then the edge intersects ∂D transversely, and
- (iii) $\Gamma \cap D \neq \emptyset$.

Let Γ be a chart. A tangle $(\Gamma \cap D, D)$ is said to be *admissible* provided that for any label k if an edge e of Γ_k intersects ∂D , then it is a normal edge and each connected component of $e \cap D$ contains a white vertex.

Let Γ be a chart, and m a label of Γ . An admissible tangle $(\Gamma \cap D, D)$ is called an *IO-tangle of label m* if there exist two distinct pseudo paths T_α, T_β in $\Gamma_m \cap D$ satisfying the following four conditions (see Fig. 4):

- (i) $T_\alpha \cap \partial D = \partial T_\alpha, T_\beta \cap \partial D = \partial T_\beta$, and $\partial T_\alpha = \partial T_\beta$.
- (ii) The intersection $T_\alpha \cap T_\beta$ consists of mutually disjoint arcs L_0, L_1, \dots, L_s ($s \geq 1$) and the closure of $(T_\alpha \cup T_\beta) - (T_\alpha \cap T_\beta)$ bounds mutually disjoint disks E_1, E_2, \dots, E_s in $\text{Int} D$ such that for each $i = 1, 2, \dots, s$ and $j = 0, 1, \dots, s$,
$$E_i \cap L_j = \begin{cases} \text{one point} & \text{if } j = i - 1 \text{ or } i, \\ \emptyset & \text{otherwise.} \end{cases}$$
- (iii) Let $X = D - (\cup_{i=1}^s E_i \cup T_\alpha \cup T_\beta)$. Then
 - (a) X consists of two connected components each of whose closures is a disk,
 - (b) one of the two disks is a side-disk of T_α , say Δ_α , and the other is a side-disk of T_β , say Δ_β , and
 - (c) $Cl(X \cap \Gamma_m)$ consists of terminal edges.

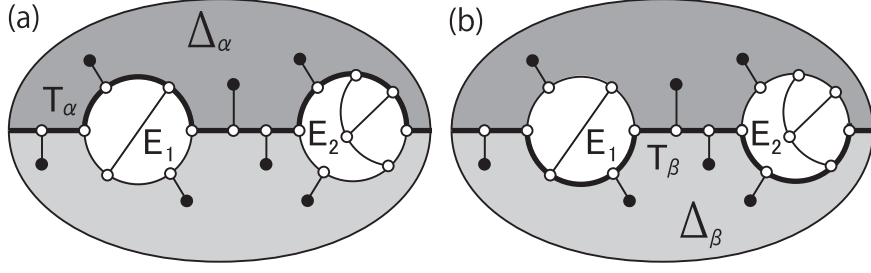


Figure 4: (a) The thick line is the pseudo path T_α . (b) The thick line is the pseudo path T_β .

- (iv) One of the pseudo paths T_α, T_β is an inward pseudo path with respect to Δ_α or Δ_β , and the other is an outward pseudo path with respect to Δ_α or Δ_β .

Theorem 1.2 *If $(\Gamma \cap D, D)$ is an admissible tangle in a minimal chart Γ such that*

- (a) $\Gamma \cap D \subset \Gamma_m \cup \Gamma_k$ for some labels m, k with $|m - k| = 1$,
- (b) $\Gamma_m \cap \partial D$ consists of exactly two points, and
- (c) $\Gamma_m \cap D$ contains a cycle,

then the tangle $(\Gamma \cap D, D)$ is an IO-tangle of label m .

The theorem above determines the structure of minimal charts with two crossings ([16] and [17]) and give us an enumeration of the charts with two crossings corresponding to 2-bridge links in \mathbb{R}^3 . If Γ is a minimal n -chart with two crossings, then there exist two labels $1 \leq \alpha < \beta \leq n - 1$ such that $\Gamma_\alpha \cap \Gamma_\beta$ consists of two crossings. Take a regular neighborhood N of $\Gamma_\alpha \cap \Gamma_\beta$. By moving free edges and simple hoops into $\text{Int}N$, there exist disjoint four disks D_1, D_2, D_3, D_4 in the complement of $\text{Int}N$ with $\Gamma_\alpha \subset D_1 \cup D_3 \cup N$ and $\Gamma_\beta \subset D_2 \cup D_4 \cup N$. Moreover for $i = 1, 3$ (resp. $i = 2, 4$), if $\Gamma_\alpha \cap D_i$ (resp. $\Gamma_\beta \cap D_i$) contains a cycle, then $(\Gamma \cap D_i, D_i)$ is an IO-tangle of label α (resp. label β) (see Fig. 5(a)). The complement of $\text{Int}(N \cup \bigcup_{i=1}^4 D_i)$ consists of disjoint four disks D'_1, D'_2, D'_3, D'_4 . We can show that for each $i = 1, 2, 3, 4$ the tangle $(\Gamma \cap D'_i, D'_i)$ is a tangle with $\Gamma \cap D'_i \subset \Gamma_{\alpha+1} \cup \Gamma_{\alpha+2} \cup \cdots \cup \Gamma_{\beta-1}$ as shown in Fig. 5(b) (see [16]). As important results, from the enumeration we can calculate the fundamental group of the exterior of the surface-link represented by Γ , and the braid monodromy of the surface braid represented by Γ .

Let Γ be a chart. An admissible tangle $(\Gamma \cap D, D)$ is called an *NS-tangle of label m* (new significant tangle) provided that

- (i) if $i \neq m$, then $\Gamma_i \cap \partial D$ is at most one point,

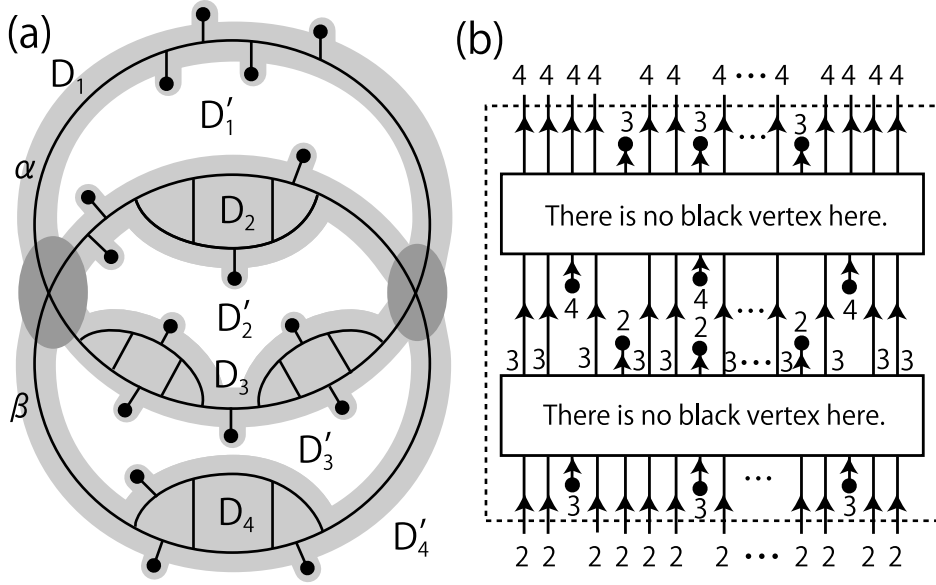


Figure 5: (a) The subgraph $\Gamma_\alpha \cup \Gamma_\beta$. (b) A tangle $(\Gamma \cap D'_i, D'_i)$ with $\Gamma \cap D'_i \subset \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ for the case $\alpha = 1$ and $\beta = 5$.

- (ii) $\Gamma \cap D$ contains at least one white vertex, and
- (iii) for each label i , the intersection $\Gamma_i \cap D$ contains at most one crossing.

Theorem 1.3 *In a minimal chart, there does not exist an NS-tangle of any label.*

The above theorem is an extended result of Theorem 3.5 in [12], and does a significant job for the classification of charts from the view point of the number of crossings.

In this paper, we denote the closure, the interior, and the boundary of (...) by $Cl(...)$, $Int(...)$, $\partial(...)$ respectively.

2 Preliminaries

Let n be a positive integer. An n -chart is an oriented labeled graph in a disk, which may be empty or have closed edges without vertices, called *hoops*, satisfying the following four conditions (see Fig. 6):

- (i) Every vertex has degree 1, 4, or 6.
- (ii) The labels of edges are in $\{1, 2, \dots, n-1\}$.
- (iii) In a small neighborhood of each vertex of degree 6, there are six short arcs, three consecutive arcs are oriented inward and the other three are outward, and these six are labeled i and $i+1$ alternately for some i ,

where the orientation and label of each arc are inherited from the edge containing the arc.

- (iv) For each vertex of degree 4, diagonal edges have the same label and are oriented coherently, and the labels i and j of the diagonals satisfy $|i - j| > 1$.

We call a vertex of degree 1 a *black vertex*, a vertex of degree 4 a *crossing*, and a vertex of degree 6 a *white vertex* respectively. Among six short arcs in a small neighborhood of a white vertex, a central arc of each three consecutive arcs oriented inward (resp. outward) is called a *middle arc* at the white vertex (see Fig. 6(c)). For each white vertex v , there are two middle arcs at v in a small neighborhood of the white vertex v .

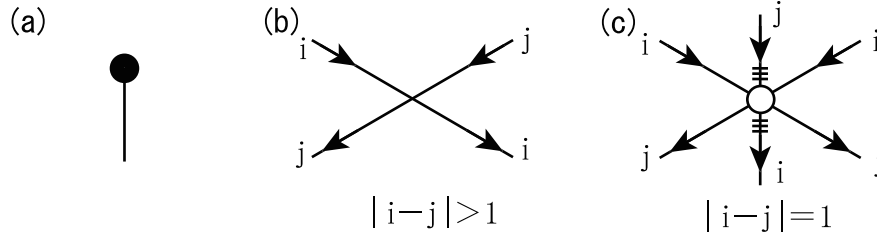


Figure 6: (a) A black vertex. (b) A crossing. (c) A white vertex. Each arc with three transversal short arcs is a middle arc at the white vertex.

Remark 2.1 Let Γ be a chart. Let $\gamma_1, \gamma_2, \dots, \gamma_6$ be six arcs around a white vertex w lying clockwise in this order. Then

- (1) for each $i = 1, 2, \dots, 6$, one of the two arcs γ_i, γ_{i+1} is not a middle arc at w , and
- (2) for each $i = 1, 2$, one of the three arcs $\gamma_i, \gamma_{i+2}, \gamma_{i+4}$ of the same label is middle at w but the others are not middle at w ,

where $\gamma_{j+6} = \gamma_j$ for each integer j .

Now *C-moves* are local modifications of charts (see [1], [6] and [19] for the precise definition). We often use C-moves shown in Fig. 7. As one of C-moves, Kamada originally defined CI-moves as follows (A C-I-M2 move and a C-I-M3 move as shown in Fig. 7 are special cases of CI-moves): A chart Γ is obtained from a chart Γ' in a disk D^2 by a *CI-move*, if there exists a disk E in D^2 such that

- (i) the two charts Γ and Γ' intersect the boundary of E transversely or do not intersect the boundary of E ,
- (ii) $\Gamma \cap E^c = \Gamma' \cap E^c$, and

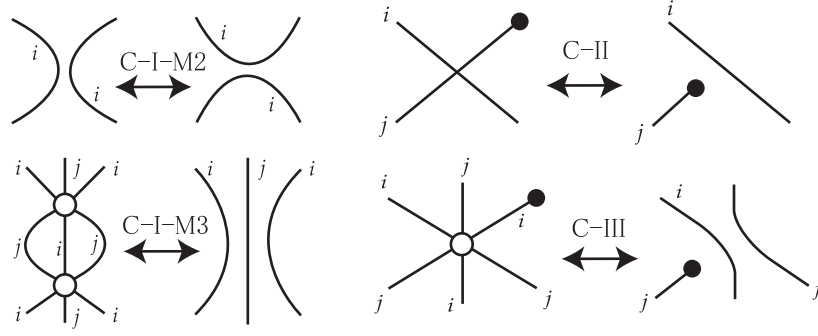


Figure 7: For the C-III move, the edge containing the black vertex does not contain a middle arc at a white vertex in the left figure.

(iii) neither $\Gamma \cap E$ nor $\Gamma' \cap E$ contains a black vertex,

where E^c is the complement of E in the disk D^2 .

Let Γ be a chart, and m a label of Γ . A closed edge of Γ_m is called a *ring* if it contains a crossing but does not contain a white vertex nor a black vertex. A *hoop* is a closed edge of Γ without vertices (hence without crossings, neither). An edge of Γ or Γ_m is called a *free edge* if it contains two black vertices. Note that loops, free edges and terminal edges may contain crossings of Γ .

To make the argument simple, we assume

Assumption 1 *In this paper, all charts are contained in the 2-sphere S^2 .*

We have the special point in the 2-sphere S^2 , called *the point at infinity*, denoted by ∞ . In this paper, all charts are contained in a disk which does not contain the point at infinity ∞ .

Remark 2.2 Let Γ be a minimal chart, and m a label of Γ . Then we have the following:

- (1) Any terminal edges and free edges of Γ_m do not contain crossings. For, if do, we can eliminate the crossings by C-II moves.
- (2) Any terminal edge of Γ contains a middle arc at its white vertex. For, if not, we can eliminate the white vertex by a C-III move.
- (3) Each complementary domain of any ring must contain at least one white vertex (cf. [8, Assumption 4]). For, suppose that there exists a ring C such that a complementary domain bounded by C does not contain any white vertices. By (1), the ring C does not intersect any terminal edge. Thus the complementary domain does not intersect any terminal edge. By C-I-M2 moves, we can move free edges out from the domain. Hence the domain does not contain any black vertices.

By a CI-move, we can eliminate the ring C . The number of crossings diminishes. This contradicts that the chart is minimal. Therefore each complementary domain of any ring must contain at least one white vertex.

A hoop is said to be *simple* if one of the complementary domains of the hoop does not contain any white vertices.

We can assume that any minimal chart Γ satisfies the following assumption [8],[11]:

Assumption 2 *All free edges and simple hoops in Γ are moved into a small neighborhood U_∞ of the point at infinity ∞ . Hence we can assume that Γ does not contain free edges nor simple hoops, otherwise mentioned.*

For, this can be done by C-I-M2 moves.

3 Admissible consecutive triplets

Let e_1 be a terminal edge of a chart Γ . A triplet (e_1, e_2, e_3) of mutually different edges of Γ is called a *consecutive triplet* if there exists an open disk U and white vertices w_1, w_2 (possibly $w_1 = w_2$) such that (see Fig. 8)

- (i) $U \cap \Gamma = \emptyset$,
- (ii) $e_3^* = e_3 \cap Cl(U)$ is an arc,
- (iii) $Cl(U) \cap \Gamma = e_1 \cup e_2 \cup e_3^*$,
- (iv) $\partial e_2 = \{w_1, w_2\}$, and
- (v) $w_1 \in e_1$ and $w_2 \in e_3^*$.

If the label of e_3 is different from the one of e_1 then the consecutive triplet is said to be *admissible*.

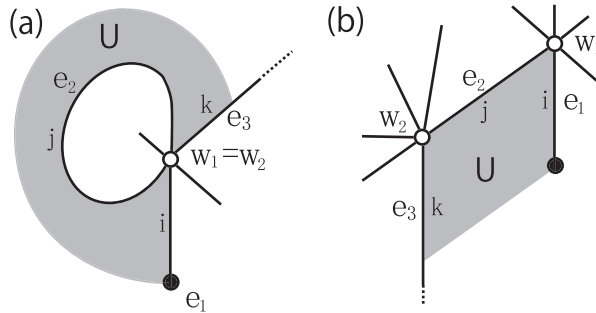


Figure 8:

Lemma 3.1 ([11, Lemma 1.1]) [**Consecutive Triplet Lemma**] *Any consecutive triplet in a minimal chart is admissible.*

The above lemma was proven by the maximality of bigons in a minimal chart. We shall prove in Section 12. In the proof of Lemma 3.3, we must be careful of the next remark.

Remark 3.2 Let w be the white vertex of a loop. In a small neighborhood of the white vertex w , the loop contains two short arcs γ, γ' with $\gamma \cap \gamma' = w$. One of the two arcs γ, γ' is a middle arc at w , but the other is not a middle arc at w .

Let G be a subset in S^2 . A complementary domain of G is said to be *finite* if it does not contain the point at infinity ∞ .

Lemma 3.3 *Let Γ be a minimal chart, and m a label of Γ . Let U be a finite complementary domain of Γ_m with $\Gamma \cap U \subset \Gamma_{m-1} \cup \Gamma_{m+1}$. Suppose that U does not contain any crossing. Then we have the following:*

- (a) *The component U does not contain any white vertex.*
- (b) *$Cl(U)$ does not contain any terminal edge of label m .*
- (c) *If U is an open disk and if $Cl(U)$ contains a white vertex in Γ_m , then there exist at least two middle arcs of label $m \pm 1$ in $Cl(U)$.*

PROOF. Since there is no white vertex contained in $\Gamma_{m-1} \cap \Gamma_{m+1}$, we have Statement (a).

For Statement (b). Suppose that $Cl(U)$ contains a terminal edge e_1 of label m . Let v_1 be the white vertex of e_1 . Then $v_1 \in \partial U$ by Lemma 3.3(a). Let e_2 be an edge of label $m \pm 1$ with $e_2 \ni v_1$ and $e_2 \cap U \neq \emptyset$. The edge e_2 is not a terminal edge. For, if e_2 is a terminal edge, then by Remark 2.1(1) one of terminal edges e_1 and e_2 does not contain a middle arc at v_1 . This contradicts Remark 2.2(2).

If e_2 is not a loop, then there exists an edge e_3 of label m in ∂U such that (e_1, e_2, e_3) is not an admissible consecutive triplet (see Fig. 9(a)). This contradicts Lemma 3.1.

Suppose that e_2 is a loop. In a regular neighborhood N of v_1 , the edge e_2 contains two short arcs γ_1, γ_2 with $\gamma_1 \cap \gamma_2 = v_1$. Now e_1 contains a short arc γ_3 in N with $\gamma_3 \ni v_1$. If $\gamma_1, \gamma_2, \gamma_3$ are not consecutive around v_1 (see Fig. 9(b)), then $v_1 \in \text{Int}(Cl(U))$. Hence again there exists an edge e_3 of label m in ∂U such that (e_1, e_2, e_3) is not an admissible consecutive triplet (see Fig. 9(b)). This contradicts Lemma 3.1.

Suppose that $\gamma_1, \gamma_2, \gamma_3$ are consecutive around v_1 . Then $\gamma_1, \gamma_3, \gamma_2$ are consecutive arcs situated around v_1 in this order (see Fig. 9(c)). By Remark 2.2(2), the arc γ_3 is middle at v_1 . Since e_2 is a loop, by Remark 3.2

one of γ_1, γ_2 is middle at v_1 , say γ_1 . Then the two consecutive arcs γ_1, γ_3 are middle at v_1 . This contradicts Remark 2.1(1). Hence Statement (b) holds.

For Statement (c). Let $W = \{v \mid v \text{ is a white vertex contained in a middle arc of label } m \pm 1 \text{ in } Cl(U)\}$. Suppose $W = \emptyset$. By the assumption, there exists a white vertex w in $Cl(U)$. Then $w \in \partial U$ by Lemma 3.3(a). Let e be an edge of label $m \pm 1$ containing the white vertex w and intersecting U . Since $W = \emptyset$, the edge e does not contain a middle arc at w . By Remark 2.2(2), the edge e is not a terminal edge.

We claim that the edge e is not a loop. For, suppose that the edge e is a loop. Since there is no crossing in $\Gamma_{m \pm 1} \cap \Gamma_m$, the edge e must be contained in $Cl(U)$. By Remark 3.2, the edge e contains a middle arc at w . Thus $w \in W$. This contradicts $W = \emptyset$. Hence the edge e is not a loop.

Thus the edge e contains two white vertices in ∂U , say $w_1 = w$ and w_2 . Then for $i = 1, 2$, there exist two edges e_{i1} and e_{i2} of label m in ∂U containing w_i . Go around ∂U starting from e_{11} and next pass through e_{12} and so on. Without loss of generality we can assume that e_{11} is oriented inward at w_1 and that we pass $e_{11}, e_{12}, \dots, e_{21}, e_{22}$ in this order. Note that when we go around ∂U (see Fig. 9(d)),

(1) the orientation of the edge in ∂U changes at a vertex in W . Since $W = \emptyset$, (1) implies that the edge e_{12} is oriented outward at w_1 , the edge e_{21} is oriented inward at w_2 , and the edge e_{22} is oriented outward at w_2 (see Fig. 9(e)). Thus applying two C-I-M2 moves between e_{11} and e_{22} and between e_{12} and e_{21} (see Fig. 9(f)), we can eliminate the two white vertices w_1 and w_2 by a C-I-M3 move. This contradicts the fact that the chart Γ is minimal. Thus $W \neq \emptyset$. Again (1) implies that the set W consists of exactly even number of vertices. Thus there exist at least two middle arcs of label $m \pm 1$ in $Cl(U)$. \square

4 A proof of Theorem 1.1

Let Γ be a chart, and m a label of Γ . A simple closed curve in Γ_m is called a *cycle of label m* . We consider hoops, rings and loops as cycles.

Let C be a simple closed curve in $S^2 - \infty$. Then C divides S^2 into two disks. One of the two disks does not contain ∞ , say D . The disk D is called the *disk bounded by C* . We also say that *the curve C bounds the disk D* .

Let Γ be a chart. Let C be a cycle of label m in Γ bounding a disk E . We define

$$\begin{aligned} \mathcal{O}(C) &= \{w \in C \mid w \text{ is a white vertex of an outside edge of } \Gamma_m \text{ for } C\}, \\ \mathcal{O}_M(C) &= \{w \in \mathcal{O}(C) \mid \text{there exists an outside edge of } \Gamma_m \text{ for } C \text{ middle at } w\}, \\ \mathcal{I}(C) &= \{w \in C \mid w \text{ is a white vertex of an inside edge of } \Gamma_m \text{ for } C\}, \\ \mathcal{I}_M(C) &= \{w \in \mathcal{I}(C) \mid \text{there exists an inside edge of } \Gamma_m \text{ for } C \text{ middle at } w\}, \\ \mathcal{W}(C) &= \{w \mid w \text{ is a white vertex in } C\}, \\ \mathcal{W}(E) &= \{w \mid w \text{ is a white vertex in } E\}, \end{aligned}$$

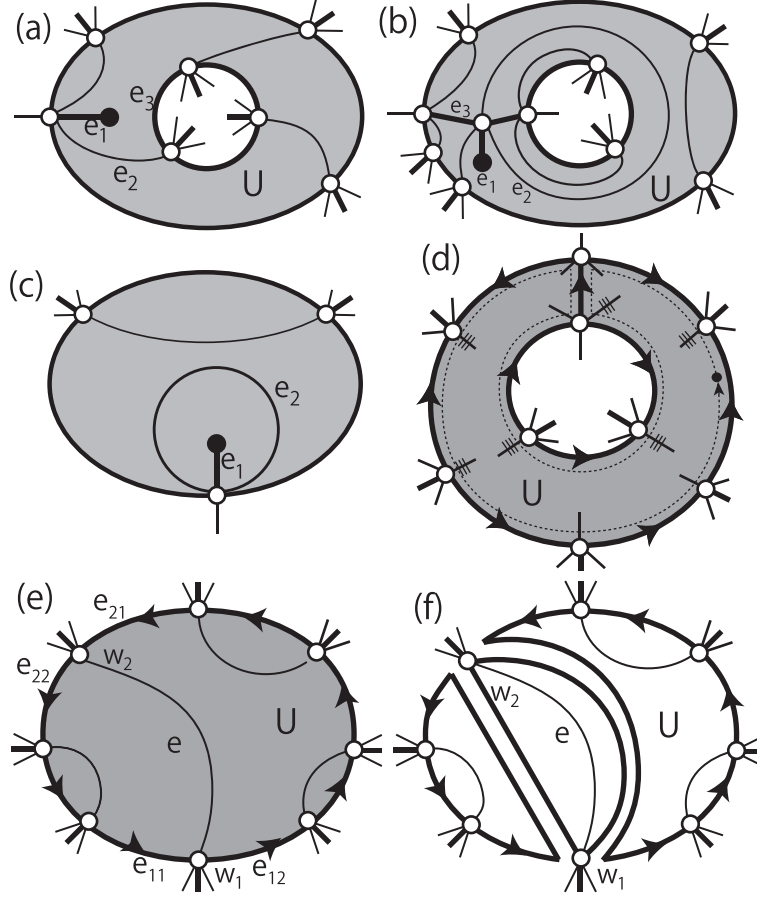


Figure 9: Each gray region is a finite complementary domain of Γ_m . The thick lines are of label m . Each arc with three transversal short arcs is a middle arc.

$$\mathcal{W}_0(E) = \{w \mid w \text{ is a white vertex in } \text{Int}E\}.$$

Remark 4.1 Let C be a cycle of label m in a chart bounding a disk E . Then we have the following:

- (1) Let w be a white vertex in $\mathcal{O}(C)$. Then the vertex w is in $\mathcal{O}_M(C)$ if and only if there exists an arc of label $m \pm 1$ in E middle at the vertex w .
- (2) Let w be a white vertex in $\mathcal{I}(C)$. Then the vertex w is *not* in $\mathcal{I}_M(C)$ if and only if there exists an arc of label $m \pm 1$ in E middle at the vertex w .
- (3) Let w be a white vertex in $\mathcal{I}(C)$. Then the vertex w is in $\mathcal{I}_M(C)$ if and only if there exists an arc of label $m \pm 1$ in $Cl(S^2 - E)$ middle at the vertex w .
- (4) The set $\mathcal{W}(C)$ splits into disjoint subsets $\mathcal{O}(C)$ and $\mathcal{I}(C)$.

- (5) The set $\mathcal{W}(E)$ splits into three mutually disjoint subsets $\mathcal{W}_0(E)$, $\mathcal{O}(C)$ and $\mathcal{I}(C)$.

Lemma 4.2 *Let Γ be a minimal chart. Let E be a three-color disk bounded by a cycle of label m . If $\Gamma_m \cap E$ is connected, then E contains neither hoop nor ring.*

PROOF. Suppose that E contains a hoop, say C . By Assumption 2, the hoop C is not simple. Thus there exists a white vertex v in the interior of the disk bounded by the hoop C . Any white vertex in E is in $\Gamma_m \cap \Gamma_{m-1}$ or $\Gamma_m \cap \Gamma_{m+1}$. Thus $\Gamma_m \cap E$ contains at least two connected components; one containing the vertex v , and the other containing ∂E . Hence $\Gamma_m \cap E$ is not connected. This is a contradiction. Thus E does not contain any hoop.

Since any three-color disk does not contain any crossing, the disk E does not contain any ring. \square

Let E be a three-color disk bounded by a cycle of label m in a chart Γ . We define

$$\mathcal{M}(E) = \{v \in \mathcal{W}(E) \mid \text{there exists an arc of label } m \pm 1 \text{ in } E \text{ middle at } v\}.$$

Lemma 4.3 *Let Γ be a minimal chart. Let E be a three-color disk bounded by a cycle C of label m . Then we have*

$$|\mathcal{M}(E)| = |\mathcal{W}_0(E)| + |\mathcal{O}_M(C)| + |\mathcal{I}(C)| - |\mathcal{I}_M(C)|.$$

Proof. By the definition of $\mathcal{M}(E)$, we have $\mathcal{M}(E) \subset \mathcal{W}(E)$. By Remark 4.1(4) and (5), the set $\mathcal{W}(E)$ is the disjoint union of $\mathcal{W}_0(E)$ and $\mathcal{W}(C)$. Thus $\mathcal{M}(E) - \mathcal{W}_0(E) = \mathcal{M}(E) \cap \mathcal{W}(C)$. Since $\mathcal{W}(C)$ is the disjoint union of $\mathcal{O}(C)$ and $\mathcal{I}(C)$ by Remark 4.1(4), we have

$$\begin{aligned} |\mathcal{M}(E) - \mathcal{W}_0(E)| &= |\mathcal{M}(E) \cap \mathcal{W}(C)| \\ &= |\mathcal{M}(E) \cap (\mathcal{O}(C) \cup \mathcal{I}(C))| \\ &= |(\mathcal{M}(E) \cap \mathcal{O}(C)) \cup (\mathcal{M}(E) \cap \mathcal{I}(C))| \\ &= |\mathcal{M}(E) \cap \mathcal{O}(C)| + |\mathcal{M}(E) \cap \mathcal{I}(C)|. \end{aligned}$$

Now by Remark 4.1(1)

$$\begin{aligned} \mathcal{M}(E) \cap \mathcal{O}(C) &= \{v \in \mathcal{O}(C) \mid \text{there exists an arc of } m \pm 1 \text{ in } E \text{ middle at } v\} \\ &= \mathcal{O}_M(C). \end{aligned}$$

Also by Remark 4.1(2)

$$\begin{aligned} \mathcal{M}(E) \cap \mathcal{I}(C) &= \{v \in \mathcal{I}(C) \mid \text{there exists an arc of } m \pm 1 \text{ in } E \text{ middle at } v\} \\ &= \{v \in \mathcal{I}(C) \mid v \notin \mathcal{I}_M(C)\} \\ &= \mathcal{I}(C) - \mathcal{I}_M(C). \end{aligned}$$

Therefore

$$\begin{aligned}
|\mathcal{M}(E)| &= |\mathcal{W}_0(E)| + |\mathcal{M}(E) - \mathcal{W}_0(E)| \\
&= |\mathcal{W}_0(E)| + |\mathcal{M}(E) \cap \mathcal{O}(C)| + |\mathcal{M}(E) \cap \mathcal{I}(C)| \\
&= |\mathcal{W}_0(E)| + |\mathcal{O}_M(C)| + |\mathcal{I}(C) - \mathcal{I}_M(C)| \\
&= |\mathcal{W}_0(E)| + |\mathcal{O}_M(C)| + |\mathcal{I}(C)| - |\mathcal{I}_M(C)|. \quad \square
\end{aligned}$$

Proof of Theorem 1.1. Since the disk E is a three-color disk,

(1) E does not contain any crossing.

Since $\Gamma_m \cap E$ is connected,

(2) E does not contain a hoop nor a ring by Lemma 4.2.

Thus

(3) for each connected component of $E - \Gamma_m$, its closure contains a white vertex.

Hence we can allow us to apply Lemma 3.3(b). Thus there does not exist any terminal edge of label m in the closure of each connected component of $E - \Gamma_m$. Hence all the vertices of Γ_m in E are white vertices. Let \mathcal{V} be the number of white vertices of Γ_m in E , and \mathcal{E} the number of edges of Γ_m in E , and \mathcal{F} the number of connected components of $E - \Gamma_m$. Since E is a three-color disk bounded by a cycle of label m , all the white vertices in E are white vertices of Γ_m . Thus we have

(4) $\mathcal{V} = |\mathcal{W}(E)|$.

Since the intersection $\Gamma_m \cap E$ is connected,

(5) each connected component of $E - \Gamma_m$ is an open disk,

and since E is a disk, we have the equation $\mathcal{V} - \mathcal{E} + \mathcal{F} = 1$ by Euler formula.

Claim 1. $\mathcal{V} = |\mathcal{W}_0(E)| + |\mathcal{O}(C)| + |\mathcal{I}(C)|$ and $2\mathcal{F} = 2 + \mathcal{V} - |\mathcal{O}(C)|$.

Proof of Claim 1. By Remark 4.1(5), the set $\mathcal{W}(E)$ splits into three mutually disjoint subsets $\mathcal{W}_0(E)$, $\mathcal{O}(C)$ and $\mathcal{I}(C)$. Thus by (4), we have $\mathcal{V} = |\mathcal{W}_0(E)| + |\mathcal{O}(C)| + |\mathcal{I}(C)|$.

In a small neighbourhood of a white vertex in Γ_m , there are exactly three short arcs of label m intersecting each other at the white vertex. We fix the three short arcs of label m for each white vertex in Γ_m . Each edge of Γ_m in E has two short arcs of label m . Thus there are $2\mathcal{E}$ short arcs in E . On the other hand, any white vertex in $\mathcal{W}_0(E) \cup \mathcal{I}(C)$ is incident with three short arcs of label m in E , and any white vertex in $\mathcal{O}(C)$ is incident with two short arcs of label m in E . Hence we have

$$2\mathcal{E} = 3(|\mathcal{W}_0(E)| + |\mathcal{I}(C)|) + 2|\mathcal{O}(C)|.$$

Thus by the equation $\mathcal{V} = |\mathcal{W}_0(E)| + |\mathcal{O}(C)| + |\mathcal{I}(C)|$, we have

$$2\mathcal{E} = 3(|\mathcal{W}_0(E)| + |\mathcal{I}(C)| + |\mathcal{O}(C)|) - |\mathcal{O}(C)| = 3\mathcal{V} - |\mathcal{O}(C)|.$$

By using the equation $\mathcal{V} - \mathcal{E} + \mathcal{F} = 1$, we have

$$2\mathcal{F} = 2 - 2\mathcal{V} + 2\mathcal{E} = 2 - 2\mathcal{V} + (3\mathcal{V} - |\mathcal{O}(C)|) = 2 + \mathcal{V} - |\mathcal{O}(C)|.$$

This completes the proof of Claim 1.

Claim 2. $|\mathcal{M}(E)| - 2\mathcal{F} \geq 0$.

Proof of Claim 2. Let $\mathcal{M}^*(E)$ be the set of all the middle arcs of label $m \pm 1$ in E . Since each white vertex is contained in exactly one middle arc of label $m \pm 1$, we have $|\mathcal{M}^*(E)| = |\mathcal{M}(E)|$. By (1), (3) and (5), Lemma 3.3(c) assures that for each connected component U of $E - \Gamma_m$, the closure $Cl(U)$ contains at least two middle arcs of label $m \pm 1$ in $\mathcal{M}^*(E)$. Hence $|\mathcal{M}^*(E)| - 2\mathcal{F} \geq 0$. Since $|\mathcal{M}^*(E)| = |\mathcal{M}(E)|$, we have $|\mathcal{M}(E)| - 2\mathcal{F} \geq 0$. This completes the proof of Claim 2.

By the first equation of Claim 1, we have $|\mathcal{W}_0(E)| = \mathcal{V} - |\mathcal{O}(C)| - |\mathcal{I}(C)|$. Thus by Lemma 4.3 we have

$$\begin{aligned} |\mathcal{M}(E)| &= |\mathcal{W}_0(E)| + |\mathcal{O}_M(C)| + |\mathcal{I}(C)| - |\mathcal{I}_M(C)| \\ &= (\mathcal{V} - |\mathcal{O}(C)| - |\mathcal{I}(C)|) + |\mathcal{O}_M(C)| + |\mathcal{I}(C)| - |\mathcal{I}_M(C)| \\ &= \mathcal{V} - |\mathcal{O}(C)| + |\mathcal{O}_M(C)| - |\mathcal{I}_M(C)|. \end{aligned}$$

By the second equation of Claim 1, we have $2\mathcal{F} = 2 + \mathcal{V} - |\mathcal{O}(C)|$. Hence we have

$$\begin{aligned} |\mathcal{M}(E)| - 2\mathcal{F} &= (\mathcal{V} - |\mathcal{O}(C)| + |\mathcal{O}_M(C)| - |\mathcal{I}_M(C)|) - (2 + \mathcal{V} - |\mathcal{O}(C)|) \\ &= |\mathcal{O}_M(C)| - |\mathcal{I}_M(C)| - 2. \end{aligned}$$

Since $0 \leq |\mathcal{M}(E)| - 2\mathcal{F}$ by Claim 2, we have $0 \leq |\mathcal{O}_M(C)| - |\mathcal{I}_M(C)| - 2$. Therefore $|\mathcal{I}_M(C)| + 2 \leq |\mathcal{O}_M(C)|$. \square

Corollary 4.4 *Let Γ be a minimal chart. Let C be a cycle of label m in Γ bounding a three-color disk E . If $\Gamma_m \cap E$ is connected, then we have*

$$|\mathcal{O}_M(C)| \geq 2.$$

5 Inward paths and outward paths

Let Γ be a chart, and P_1^*, P_2^* pseudo paths of label m in Γ . The pair (P_1^*, P_2^*) is called an *I/O pair of type I* if there exist side-disks Δ_1, Δ_2 of P_1^*, P_2^* respectively (see Fig. 10) such that

- (i) P_1^*, P_2^* are I/O pseudo paths with respect to the side-disks Δ_1, Δ_2 respectively,
- (ii) $P_1^* \cup P_2^*$ is a pseudo path,
- (iii) $\Delta_1 \cup \Delta_2$ is a side-disk of $P_1^* \cup P_2^*$, and
- (iv) the intersection $P_1^* \cap P_2^*$ contains exactly one white vertex.

We denote the pseudo path $P_1^* \cup P_2^*$ by $P_1^* * P_2^*$. The union $\Delta_1 \cup \Delta_2$ is called an *associated side-disk* of $P_1^* * P_2^*$. The pair (Δ_1, Δ_2) is called an *associated side-disk pair* of the I/O pair (P_1^*, P_2^*) .

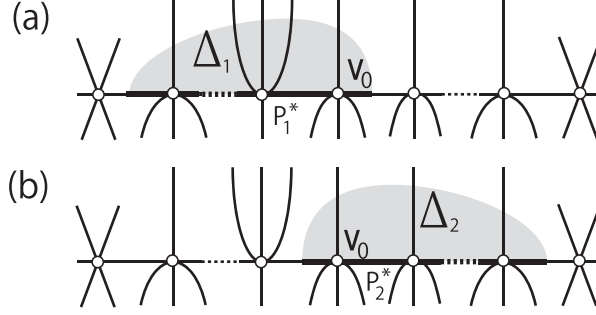


Figure 10: The thick lines are pseudo paths P_1^*, P_2^* .

Remark 5.1 Since an I/O pseudo path does not contain any crossing by the definition, any vertex in an I/O pseudo path is a white vertex.

Lemma 5.2 *Let Γ be a chart, and P_1^*, P_2^* pseudo paths of label m in Γ . If (P_1^*, P_2^*) is an I/O pair of type I, then $P_1^* * P_2^*$ is an I/O pseudo path with respect to its associated side-disk.*

PROOF. We use the notations in the definition of an I/O pair of type I. Without loss of generality we can assume that the pseudo path P_1^* is inward with respect to the side-disk Δ_1 . Then for each vertex v in P_1^* , any side-arc at v of P_1^* with respect to Δ_1 is oriented inward at v . Thus we have

- (1) for each vertex v in $P_1^* \subset P_1^* * P_2^*$, any side-arc at v of $P_1^* * P_2^*$ with respect to $\Delta_1 \cup \Delta_2$ is oriented inward at v .

By condition (iv) for the definition of an I/O pair of type I, there exists exactly one vertex v_0 in $P_1^* \cap P_2^*$. Since the white vertex v_0 is a common vertex of P_1^* and P_2^* , any side-arc at v_0 of P_2^* with respect to Δ_2 is oriented inward at v_0 , too. Hence the I/O pseudo path P_2^* is inward with respect to Δ_2 . Thus for each vertex v in P_2^* , any side-arc at v of P_2^* with respect to Δ_2 is oriented inward at v . Hence we have

- (2) for each vertex v in $P_2^* \subset P_1^* * P_2^*$, any side-arc at v of $P_1^* * P_2^*$ with respect to $\Delta_1 \cup \Delta_2$ is oriented inward at v .

Therefore by (1) and (2), the pseudo path $P_1^* * P_2^*$ is inward with respect to $\Delta_1 \cup \Delta_2$. \square

Let P^* be a pseudo path of label m in a chart, and v_1, v_2, \dots, v_s all the vertices in P^* which are situated in this order on P^* , here some of v_1, v_2, \dots, v_s may be crossings. For each $i = 1, 2, \dots, s-1$, let e_i be the edge of label m in P^* with $\partial e_i = \{v_i, v_{i+1}\}$. Then the s -tuple (v_1, v_2, \dots, v_s) is called the *associated vertex sequence* for the pseudo path P^* , and the $(s-1)$ -tuple $(e_1, e_2, \dots, e_{s-1})$ is called the *associated edge sequence* for the pseudo path

P^* . The path $e_1 \cup e_2 \cup \dots \cup e_{s-1}$ is denoted by $L(P^*)$. Let γ_0 and γ_s be arcs in edges of label m with $\gamma_0 \ni v_1, \gamma_s \ni v_s$ and $P^* = \gamma_0 \cup L(P^*) \cup \gamma_s$. Then γ_0, γ_s are called the *end-arcs* of the pseudo path P^* .

Let Γ be a chart, and P_1^*, P_2^* pseudo paths of label m in Γ . The pair (P_1^*, P_2^*) is called an *I/O pair of type II* if there exist side-disks Δ_1, Δ_2 of P_1^*, P_2^* respectively (see Fig. 11) such that

- (i) the pseudo paths P_1^* and P_2^* are I/O pseudo paths with respect to the side-disks Δ_1, Δ_2 respectively,
- (ii) the intersection $\gamma = \Delta_1 \cap \Delta_2$ is an end-arc of P_1^* and P_2^* respectively,
- (iii) γ is middle at the white vertex v_0 in γ ,
- (iv) $P^* = Cl((P_1^* \cup P_2^*) - \gamma)$ is a pseudo path, and $P^* \cap \gamma = v_0$ and
- (v) the union $\Delta_1 \cup \Delta_2$ is a side-disk of P^* .

We denote P^* by $P_1^* * P_2^*$. The union $\Delta_1 \cup \Delta_2$ is called an *associated side-disk* of $P_1^* * P_2^*$. The pair (Δ_1, Δ_2) is called an *associated side-disk pair* of the I/O pair (P_1^*, P_2^*) .

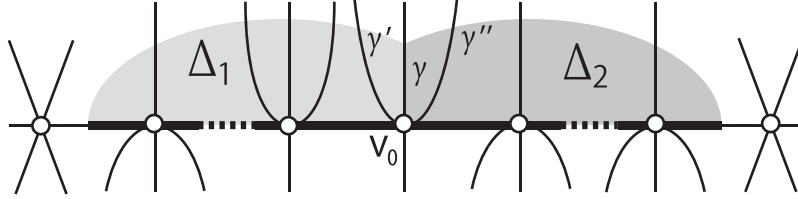


Figure 11: The thick line is a pseudo path P^* . The end-arc γ is a middle arc at v_0 . And Δ_1, Δ_2 are side-disks of P_1^*, P_2^* respectively.

Lemma 5.3 *Let Γ be a chart. Let P_1^*, P_2^* be pseudo paths of label m in Γ . If (P_1^*, P_2^*) is an I/O pair of type II, then $P_1^* * P_2^*$ is an I/O pseudo path with respect to its associated side-disk.*

PROOF. We use the notations in the definition of an I/O pair of type II. The end-arc γ is a side-arc of $P_1^* * P_2^*$ middle at the vertex v_0 . Without loss of generality we can assume that the side-arc γ is oriented inward at v_0 . Let γ' be a side-arc at v_0 of P_1^* with respect to Δ_1 , and γ'' a side-arc at v_0 of P_2^* with respect to Δ_2 (see Fig. 11). Since γ is middle at v_0 , the side-arcs γ', γ'' are oriented inward at v_0 . Thus the pseudo paths P_1^* and P_2^* are inward with respect to Δ_1 and Δ_2 respectively. Hence we have

- (1) for each vertex v in P_1^* , if $v \neq v_0$, then any side-arc at v of $P_1^* * P_2^*$ with respect to $\Delta_1 \cup \Delta_2$ is oriented inward at v , and

- (2) for each vertex v in P_2^* , if $v \neq v_0$, then any side-arc at v of $P_1^* * P_2^*$ with respect to $\Delta_1 \cup \Delta_2$ is oriented inward at v .

The three side-arcs $\gamma', \gamma, \gamma''$ are oriented inward at v_0 . Therefore by (1) and (2), the pseudo path $P_1^* * P_2^*$ is inward with respect to $\Delta_1 \cup \Delta_2$. \square

Let Γ be a chart, m a label of Γ , and $s \in \mathbb{N}$ with $s \geq 2$. Let P^* be a pseudo path of label m in Γ with a side-disk Δ , and $P_1^*, P_2^*, \dots, P_s^*$ pseudo paths of label m in Γ such that

- (i) for $i, j \in \{1, 2, \dots, s\}$, if $|i - j| > 1$, then $P_i^* \cap P_j^* = \emptyset$,
- (ii) for each $k = 1, 2, \dots, s$, $L_k = \cup_{i=1}^k L(P_i^*)$ is a path in P^* , and
- (iii) $L_s = L(P^*)$.

The s -tuple $(P_1^*, P_2^*, \dots, P_s^*)$ is called an *I/O sequence* for (P^*, Δ) if there exist side-disks $\Delta_1, \Delta_2, \dots, \Delta_s$ of $P_1^*, P_2^*, \dots, P_s^*$ respectively such that

- (i) for each $k = 1, 2, \dots, s$, $\Delta'_k = \cup_{i=1}^k \Delta_i$ is a disk in Δ ,
- (ii) for each $i = 1, 2, \dots, s - 1$, the pair (P_i^*, P_{i+1}^*) is an I/O pair of type I or type II with an associated side-disk pair (Δ_i, Δ_{i+1}) , and
- (iii) Δ'_s is a side-disk of P^* .

The s -tuple $(\Delta_1, \Delta_2, \dots, \Delta_s)$ is called an *associated side-disk sequence* of the I/O sequence $(P_1^*, P_2^*, \dots, P_s^*)$.

Remark 5.4 Let $(P_1^*, P_2^*, \dots, P_s^*)$ be an I/O sequence for (P^*, Δ) with an associated side-disk sequence $(\Delta_1, \Delta_2, \dots, \Delta_s)$. Let $Q_1^* = P_1^*$. Since the side-disk Δ of the pseudo path P^* contains all the side-disks $\Delta_1, \Delta_2, \dots, \Delta_s$ of $P_1^*, P_2^*, \dots, P_s^*$, we can inductively show that for each $i = 2, 3, \dots, s$, the pair (Q_{i-1}^*, P_i^*) is an I/O pair of type I (resp. type II) with an associated side-disk pair $(\Delta'_{i-1}, \Delta_i)$ if the pair (P_{i-1}^*, P_i^*) is an I/O pair of type I (resp. type II), and that $Q_i^* = Q_{i-1}^* * P_i^*$ is an I/O pseudo path with respect to Δ'_i by Lemma 5.2 (resp. Lemma 5.3). Hence Q_s^* is an I/O pseudo path. Therefore P^* is an I/O pseudo path. We denote the I/O pseudo path Q_s^* by $P_1^* * P_2^* * \dots * P_s^*$.

6 Monochromatic pseudo paths

Let P^* be a pseudo path of label m in a chart, and (v_1, v_2, \dots, v_s) the associated vertex sequence for P^* . Let Δ be a side-disk of P^* . The pseudo path P^* is said to be *admissible* for the side-disk Δ provided that

- (i) the two vertices v_1 and v_s are white vertices, and

Lemma 6.2 *Let Γ be a minimal chart. Let P^* be a monochromatic pseudo path of label m in Γ with the associated vertex sequence (v_1, v_2, \dots, v_s) , the associated edge sequence $(e_1, e_2, \dots, e_{s-1})$ and an associated side-arc sequence $(\gamma'_1, \gamma'_2, \dots, \gamma'_s)$. Let γ_0 and γ_s be the end-arcs of P^* with $\gamma_0 \ni v_1$ and $\gamma_s \ni v_s$. Suppose that each side-arc γ'_i ($i = 1, 2, \dots, s$) is not middle at v_i . Then we have the following:*

- (a) *If the end-arc γ_0 is oriented inward at v_1 , then each edge e_i ($i = 1, 2, \dots, s-1$) is oriented inward at v_{i+1} , and the end-arc γ_s is oriented outward at v_s .*
- (b) *If the end-arc γ_0 is oriented outward at v_1 , then each edge e_i ($i = 1, 2, \dots, s-1$) is oriented outward at v_{i+1} , and the end-arc γ_s is oriented inward at v_s .*
- (c) *The pseudo path P^* is an I/O pseudo path.*

PROOF. Set $e_0 = \gamma_0, e_s = \gamma_s$. Let v_{s+1} be the endpoint of γ_s different from v_s .

For Statement (a). Now the end-arc γ_0 is oriented inward at v_1 . Suppose that for some integer i ($1 \leq i \leq s$) e_i is oriented outward at v_{i+1} . Let $t = \min\{j \mid e_j \text{ is oriented outward at } v_{j+1}\}$. Then $t \geq 1$ and e_t is oriented outward at v_{t+1} . Thus e_t is oriented inward at v_t . But e_{t-1} is oriented inward at v_t . Hence the side-arc γ'_t is middle at v_t . This contradicts the assumption. Hence Statement (a) holds.

Similarly we can show Statement (b).

For Statement (c). We only show the case that the end-arc γ_0 is oriented inward at v_1 . Then by Lemma 6.2(a), each edge e_i ($i = 1, 2, \dots, s-1$) is oriented inward at v_{i+1} , and the end-arc γ_s is oriented outward at v_s . Suppose that the pseudo path P^* is not an I/O pseudo path. Then for some two integers i, j with $1 \leq i < j \leq s$, one of the following occurs.

- (1) The side-arc γ'_i is oriented inward at v_i , and the side-arc γ'_j is oriented outward at v_j .
- (2) The side-arc γ'_i is oriented outward at v_i , and the side-arc γ'_j is oriented inward at v_j .

Without loss of generality we can assume $j = i + 1$. We show that we can eliminate the two white vertices v_i and v_{i+1} by C-moves. For, either Case (1) or Case (2), we can apply a C-I-M2 move between the two side-arcs γ'_i and γ'_{i+1} (see Fig. 13(a) and (b)). Since e_{i-1} is oriented inward at v_i and since e_{i+1} is oriented inward at v_{i+2} , we can apply a C-I-M2 move between e_{i-1} and e_{i+1} (see Fig. 13(c)). Finally by applying a C-I-M3 move, we can eliminate the two white vertices v_i and v_{i+1} . This contradicts the fact that the chart Γ is minimal. Hence the pseudo path P^* is an I/O pseudo path. This completes the proof of Statement (c). \square

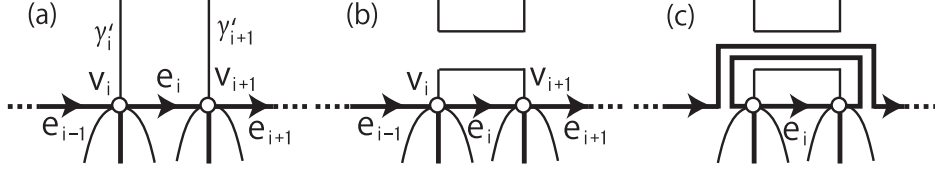


Figure 13:

Lemma 6.3 *Let Γ be a minimal chart. Let P^* be a monochromatic pseudo path of label m in Γ with the associated vertex sequence (v_1, v_2, \dots, v_s) and an associated side-arc sequence $(\gamma'_1, \gamma'_2, \dots, \gamma'_s)$. Let γ_0 and γ_s be the end-arcs of P^* with $\gamma_0 \ni v_1$ and $\gamma_s \ni v_s$. Suppose that*

- (a) *the end-arc γ_0 is middle at v_1 or the end-arc γ_s is middle at v_s , and*
- (b) *for each $i = 2, 3, \dots, s-1$, the side-arc γ'_i is not middle at v_i .*

Then the pseudo path P^ is an I/O pseudo path.*

PROOF. By (a), without loss of generality we can assume that

- (1) the end-arc γ_s is middle at v_s .

The side-arc γ'_1 is middle at v_1 or not middle at v_1 .

First, suppose that the side-arc γ'_1 is not middle at v_1 . Then by (b) and (1), we have that for each $i = 1, 2, \dots, s$, the side-arc γ'_i is not middle at v_i . Thus by Lemma 6.2(c), the pseudo path P^* is an I/O pseudo path.

Next, suppose that the side-arc γ'_1 is middle at v_1 . If $s = 1$ then our lemma is true. Thus we assume $s \geq 2$. Let $(e_1, e_2, \dots, e_{s-1})$ be the associated edge sequence for P^* . Without loss of generality we can assume that

- (2) the side-arc γ'_1 is oriented inward at v_1 .

Then the edge e_1 is oriented inward at v_1 . Thus the edge e_1 is oriented outward at v_2 . Since the end-arc γ_s is middle at v_s by (1), the side-arc γ'_s is not middle at v_s . Hence by (b), we have that for each $i = 2, 3, \dots, s$, the side-arc γ'_i is not middle at v_i . Let γ be a short arc containing v_2 in the edge e_1 . We can apply Lemma 6.2 to the monochromatic pseudo path $P_1^* = \gamma \cup e_2 \cup \dots \cup e_{s-1} \cup \gamma_s$. Then the pseudo path P_1^* is an I/O pseudo path, and the end-arc γ_s is oriented inward at v_s because e_1 is oriented outward at v_2 (i.e. the arc γ is oriented outward at v_2). Thus γ'_s is oriented inward at v_s because the end-arc γ_s is middle at v_s by (1). Hence the pseudo path P_1^* is an inward pseudo path. Thus for each $i = 2, 3, \dots, s$, the side-arc γ'_i is oriented inward at v_i . Considering (2), we have for each $i = 1, 2, \dots, s$, the side-arc γ'_i is oriented inward at v_i . Therefore the pseudo path P^* is an inward pseudo path. This completes the proof of Lemma 6.3. \square

Lemma 6.4 *Let Γ be a minimal chart. Let P^* be a monochromatic pseudo path of label m in Γ with the associated vertex sequence (v_1, v_2, \dots, v_s) and an associated side-arc sequence $(\gamma'_1, \gamma'_2, \dots, \gamma'_s)$. Suppose that the end-arcs γ_0 and γ_s of P^* are middle at v_1 and v_s respectively. Then we have the following:*

- (a) $s \geq 3$.
- (b) For some integer t with $2 \leq t \leq s - 1$, the side-arc γ'_t is middle at v_t .
- (c) If there exists an integer t with $2 \leq t \leq s - 1$ such that
 - (i) the side-arc γ'_t is middle at v_t , and
 - (ii) for each $i = 1, 2, \dots, s$ with $i \neq t$, the side-arc γ'_i is not middle at v_i ,

then the pseudo path P^ is an I/O pseudo path.*

PROOF. First, we shall prove Statement (b). Without loss of generality we can assume that the end-arc γ_0 is oriented inward at v_1 . Since γ_0 is middle at v_1 , we have

- (1) the side-arc γ'_1 is oriented inward at v_1 .

Since the end-arcs γ_0, γ_s are middle at v_1, v_s respectively, Remark 2.1(1) implies

- (2) the side-arc γ'_1 (resp. γ'_s) is not middle at v_1 (resp. v_s).

Now suppose that for each $i = 2, \dots, s - 1$ the side-arc γ'_i is not middle at v_i . Then by (2), we have that for $i = 1, 2, \dots, s$ the side-arc γ'_i is not middle at v_i . Hence by Lemma 6.2(c), the monochromatic pseudo path P^* is an I/O pseudo path. Further since the end-arc γ_0 is oriented inward at v_1 , by Lemma 6.2(a) the end-arc γ_s is oriented outward at v_s . Since γ_s is middle at v_s , we have

- (3) the side-arc γ'_s is oriented outward at v_s .

Hence the side-arc γ'_1 is oriented outward at v_1 because P^* is an I/O pseudo path. This contradicts (1). Hence for some integer t with $2 \leq t \leq s - 1$, the side-arc γ'_t is middle at v_t . Thus Statement (b) holds.

For Statement (a). By Lemma 6.4(b), we have $s - 1 \geq 2$. Thus $s \geq 3$.

For Statement (c). Let $(e_1, e_2, \dots, e_{s-1})$ be the associated edge sequence for P^* . Let γ_{t-1} and γ_t be short arcs containing v_t in the edges e_{t-1} and e_t respectively. Set $P_1^* = \gamma_0 \cup e_1 \cup e_2 \cdots \cup e_{t-1} \cup \gamma_t$, and $P_2^* = \gamma_{t-1} \cup e_t \cup \cdots \cup e_{s-1} \cup \gamma_s$. Then P_1^* and P_2^* are monochromatic pseudo paths by Remark 6.1(2). Thus P_1^* and P_2^* are I/O pseudo paths by Lemma 6.3. Thus (P_1^*, P_2^*) is an I/O pair of type I. Therefore the pseudo path P^* is an I/O pseudo path by Lemma 5.2. \square

7 Two-color disks

For a simple arc P , we denote by ∂P the set consisting of the two endpoints of P .

Let C be a cycle of label m in a chart Γ , and P a path in C with $\partial P = \{v_1, v_2\} \subset \mathcal{O}(C)$ (see Fig. 14(a)). Let E be the disk bounded by C , and N a regular neighborhood of P . For $i = 1, 2$, let e_i be the outside edge of Γ_m for C containing v_i , and γ_i the connected component of $e_i \cap N$ containing v_i (see Fig. 14(b)). Set $\hat{P} = \gamma_1 \cup P \cup \gamma_2$. Then the union \hat{P} is called the *extended pseudo path* of P .

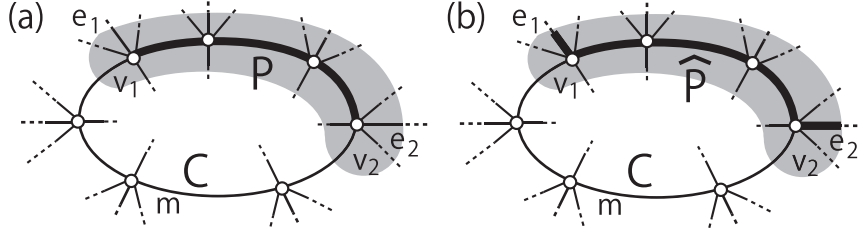


Figure 14: The vertices v_1, v_2 are in $\mathcal{O}(C)$. (a) The thick line is a path P , the gray region is a regular neighborhood N of P . (b) The thick line is the extended pseudo path \hat{P} .

Remark 7.1 Any extended pseudo path is admissible.

Let Γ be a chart, and E a disk bounded by a cycle of label m in Γ . Then the disk E is called a *two-color disk* provided that $\Gamma \cap E \subset \Gamma_m \cup \Gamma_k$ for some label k with $|m - k| = 1$. The label k is called the *secondary label* of E .

Remark 7.2 By the condition of a two-color disk, any two-color disk does not contain any crossing. Hence any two-color disk is also a three-color disk.

Theorem 7.3 *Let Γ be a minimal chart. Let C be a cycle of label m in Γ bounding a two-color disk E with $\Gamma_m \cap E$ connected. Suppose that $|\mathcal{O}(C) - \mathcal{O}_M(C)| = 2$. Let S and T be the two paths obtained from C by cutting at the two white vertices in $\mathcal{O}(C) - \mathcal{O}_M(C)$. Then we have the following:*

- (a) *Each of the paths S and T contains at least one white vertex in $\mathcal{O}_M(C)$.*
- (b) *One of the extended pseudo paths \hat{S}, \hat{T} of S, T is an inward pseudo path, and the other is an outward pseudo path.*

Let Γ be a chart. Let C be a cycle of label m or a path in a cycle of label m in Γ , and \mathcal{S} a set of white vertices in C , here we assume that $|\mathcal{S}| \geq 2$ if C is a cycle. By cutting C at all the white vertices in \mathcal{S} , the set C splits into

paths. Then the set of all the paths is called the *path decomposition* of C by \mathcal{S} , denoted by $\mathcal{P}(C; \mathcal{S})$.

We extend the above theorem as follows (hence we do not give a proof of Theorem 7.3):

Theorem 7.4 *Let Γ be a minimal chart. Let C be a cycle of label m in Γ bounding a two-color disk E with $\Gamma_m \cap E$ connected. Suppose that there are two paths S, T in the path decomposition $\mathcal{P}(C; \mathcal{O}(C) - \mathcal{O}_M(C))$ with $\mathcal{O}_M(C) \subset S \cup T$. Let $X = \{ P \mid P \in \mathcal{P}(S; \mathcal{O}_M(C) \cap S) \cup \mathcal{P}(T; \mathcal{O}_M(C) \cap T), \partial P \subset \mathcal{O}_M(C) \}$. Then we have the following:*

- (a) *Each of the paths S and T contains at least one white vertex in $\mathcal{O}_M(C)$.*
- (b) *Each path in X contains exactly one white vertex in $\mathcal{I}_M(C)$.*
- (c) *$C - (\cup_{P \in X} P)$ does not contain any white vertex in $\mathcal{I}_M(C)$.*
- (d) *The extended pseudo paths \hat{S}, \hat{T} of S, T are I/O pseudo paths.*

Lemma 7.5 *Let Γ be a chart. Let C be a cycle of label m in Γ bounding a two-color disk. Let P be a path in $\mathcal{P}(C; \mathcal{O}(C))$. Then the extended pseudo path of P is a monochromatic pseudo path.*

PROOF. Let k be the secondary label of the two-color disk. By Remark 7.2, the two-color disk does not contain any crossing, neither does P . Let v be a white vertex in $P - \partial P$. Then v is not in $\mathcal{O}(C)$ by the assumption. Hence the vertex v is in $\mathcal{I}(C)$. Thus there exists exactly one outside edge of label $m \pm 1$ at the vertex v . Since the cycle C bounds a two-color disk, the label of the outside edge is k . Therefore the extended pseudo path of P is a monochromatic pseudo path. \square

Lemma 7.6 *Let Γ be a minimal chart. Let C be a cycle of label m in Γ bounding a two-color disk E . If $\Gamma_m \cap E$ is connected, then*

- (a) *in the path decomposition $\mathcal{P}(C; \mathcal{O}_M(C))$ there exist at least two paths each of which contains a white vertex in $\mathcal{O}(C) - \mathcal{O}_M(C)$,*
- (b) *$|\mathcal{O}(C) - \mathcal{O}_M(C)| \geq 2$, and*
- (c) *in the path decomposition $\mathcal{P}(C; \mathcal{O}(C) - \mathcal{O}_M(C))$ there exist at least two paths each of which contains a white vertex in $\mathcal{O}_M(C)$.*

PROOF. By Remark 7.2, the two-color disk is also a three-color disk. Let $s = |\mathcal{O}_M(C)|$. Then $s \geq 2$ by Corollary 4.4. There are s paths in the path decomposition $\mathcal{P}(C; \mathcal{O}_M(C))$.

For Statement (a). Suppose that in $\mathcal{P}(C; \mathcal{O}_M(C))$ there exists at most one path containing a white vertex in $\mathcal{O}(C) - \mathcal{O}_M(C)$. Then in $\mathcal{P}(C; \mathcal{O}_M(C))$

there are $s - 1$ paths each of which does not contain any vertices in $\mathcal{O}(C)$ except its end vertices. By Lemma 7.5, the extended pseudo paths of the $s - 1$ paths are monochromatic pseudo paths. Let k be the secondary label of the two-color disk E . By Lemma 6.4(b), each of the pseudo paths has a side-arc of label k middle at a vertex in the pseudo path, namely each of the paths contains a vertex in $\mathcal{I}_M(C)$ by Remark 4.1(3). Thus $|\mathcal{I}_M(C)| \geq s - 1$.

By Remark 7.2, the two-color disk is a three-color disk. Thus by Theorem 1.1

$$s + 1 = (s - 1) + 2 \leq |\mathcal{I}_M(C)| + 2 \leq |\mathcal{O}_M(C)| = s.$$

This is a contradiction. Therefore Statement (a) holds.

Now Statement (b) is a direct result of Lemma 7.6(a).

For Statement (c). By Lemma 7.6(b), we have $|\mathcal{O}(C) - \mathcal{O}_M(C)| \geq 2$. Thus we can consider the path decomposition $\mathcal{P}(C; \mathcal{O}(C) - \mathcal{O}_M(C))$. Suppose that in the path decomposition $\mathcal{P}(C; \mathcal{O}(C) - \mathcal{O}_M(C))$ there exists a path P containing all the white vertices in $\mathcal{O}_M(C)$ (see Fig. 15). Then all the paths in $\mathcal{P}(C; \mathcal{O}_M(C))$ are contained in P except one, say Q . Thus $P \cup Q = C$. Since $P \in \mathcal{P}(C; \mathcal{O}(C) - \mathcal{O}_M(C))$ implies $\text{Int}P \cap (\mathcal{O}(C) - \mathcal{O}_M(C)) = \emptyset$, we have $\mathcal{O}(C) - \mathcal{O}_M(C) \subset Q$. This contradicts Lemma 7.6(a). \square

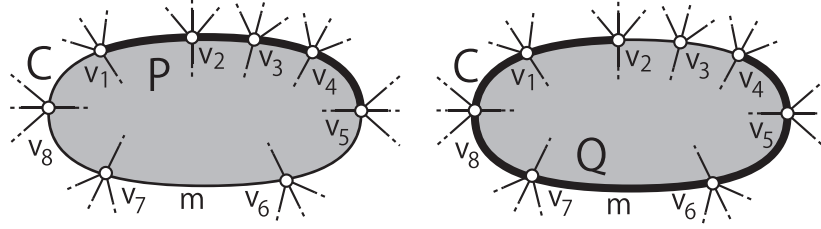


Figure 15: An example of a path P in a cycle C of label m with $\mathcal{O}_M(C) \subset P$. The thick lines are paths P and Q with $\partial P = \{v_1, v_5\}$ and $\partial Q = \{v_2, v_4\}$, $\mathcal{O}(C) = \{v_1, v_2, \dots, v_8\}$ and $\mathcal{O}_M(C) = \{v_2, v_3, v_4\}$.

For a simple arc P , we define $\text{Int}P = P - \partial P$.

Proof of Theorem 7.4. Let $s = |S \cap \mathcal{O}_M(C)|$, and $t = |T \cap \mathcal{O}_M(C)|$. Now Lemma 7.6(c) implies $s \geq 1, t \geq 1$. Thus Statement (a) holds.

For Statement (b). Let $\mathcal{S} = \mathcal{P}(S; \mathcal{O}_M(C) \cap S) = \{S_1, S_2, \dots, S_{s+1}\}$, and $\mathcal{T} = \mathcal{P}(T; \mathcal{O}_M(C) \cap T) = \{T_1, T_2, \dots, T_{t+1}\}$. Since $S, T \in \mathcal{P}(C; \mathcal{O}(C) - \mathcal{O}_M(C))$, we have

$$(1) \quad \{S_1, S_2, \dots, S_{s+1}, T_1, T_2, \dots, T_{t+1}\} \subset \mathcal{P}(C; \mathcal{O}(C)).$$

Since $S, T \in \mathcal{P}(C; \mathcal{O}(C) - \mathcal{O}_M(C))$, without loss of generality we can assume that

$$(2) \quad \text{each of } S_1, S_{s+1}, T_1, T_{t+1} \text{ contains a vertex in } \mathcal{O}(C) - \mathcal{O}_M(C) \text{ and a vertex in } \mathcal{O}_M(C), \text{ and}$$

- (3) for $i = 2, 3, \dots, s$ and $j = 2, 3, \dots, t$, we have $\partial S_i \subset \mathcal{O}_M(C)$ and $\partial T_j \subset \mathcal{O}_M(C)$.

Set $X = \{S_2, S_3, \dots, S_s, T_2, T_3, \dots, T_t\}$. Since $\mathcal{O}_M(C) \subset S \cup T$, we have

- (4) $s + t = |\mathcal{O}_M(C)|$.

Let k be the secondary label of the two-color disk. Now by Remark 4.1(3)

- (5) for each path P in $\mathcal{S} \cup \mathcal{T}$, a side-arc of label k of the extended pseudo path \widehat{P} of P is middle at a vertex v in $\text{Int}P$ if and only if $v \in \mathcal{I}_M(C)$.

For each path $P \in X$, (1) implies that the extended pseudo path \widehat{P} of P is a monochromatic pseudo path by Lemma 7.5. Further (3) assures that we can apply Lemma 6.4(b) to \widehat{P} so that the extended pseudo path \widehat{P} has a side-arc of label k middle at a vertex in $\text{Int}P$. Namely by (5)

- (6) for each $P \in X$, the set $\text{Int}P$ contains at least one vertex in $\mathcal{I}_M(C)$.

For each $i = 1, 2, \dots, s + 1$ and $j = 1, 2, \dots, t + 1$, let

$$\sigma_i = |S_i \cap \mathcal{I}_M(C)|, \quad \tau_j = |T_j \cap \mathcal{I}_M(C)|.$$

Then (6) implies

- (7) for each $i = 2, 3, \dots, s$ and $j = 2, 3, \dots, t$, we have $1 \leq \sigma_i$ and $1 \leq \tau_j$.

Claim. $\sigma_1 = \sigma_{s+1} = \tau_1 = \tau_{t+1} = 0$, and $\sigma_2 = \sigma_3 = \dots = \sigma_s = \tau_2 = \tau_3 = \dots = \tau_t = 1$.

For, if not, then by (7), we have

$\sigma_1 > 0$, or $\sigma_{s+1} > 0$, or $\tau_1 > 0$, or $\tau_{t+1} > 0$, or

$\sigma_i > 1$ for some $2 \leq i \leq s$, or $\tau_j > 1$ for some $2 \leq j \leq t$.

Thus we have

$$\begin{aligned} |\mathcal{I}_M(C)| &\geq \sigma_1 + (\sigma_2 + \sigma_3 + \dots + \sigma_s) + \sigma_{s+1} + \tau_1 + (\tau_2 + \tau_3 + \dots + \tau_t) + \tau_{t+1} \\ (8) \quad &> 0 + (1 + 1 + \dots + 1) + 0 + 0 + (1 + 1 + \dots + 1) + 0 \\ &= (s - 1) + (t - 1) = s + t - 2. \end{aligned}$$

Using the equation (4) and the inequation (8), Theorem 1.1 implies

$$s + t = (s + t - 2) + 2 < |\mathcal{I}_M(C)| + 2 \leq |\mathcal{O}_M(C)| = s + t.$$

This is a contradiction. Therefore Claim holds.

Now Statement (b) follows from Claim.

For Statement (c). If $C - (\cup_{P \in X} P)$ contains a white vertex in $\mathcal{I}_M(C)$, then Theorem 7.4(b) implies the inequation $|\mathcal{I}_M(C)| > s + t - 2$. Thus again we have the same contradiction. Thus Statement (c) holds.

For Statement (d). Without loss of generality we can assume that S_1, S_2, \dots, S_{s+1} are situated on S in this order. Now (1) implies that for each $i = 1, 2, \dots, s + 1$, the extended pseudo path \widehat{S}_i of S_i is a monochromatic pseudo path by Lemma 7.5. Thus Theorem 7.4(b) implies that the extended pseudo paths

$\widehat{S}_2, \widehat{S}_3, \dots, \widehat{S}_s$ are I/O pseudo paths by Lemma 6.4(c). Further, by Theorem 7.4(c) and (5), for any vertex $v \in \text{Int}S_1$ (resp. $v \in \text{Int}S_{s+1}$) any side-arc at v of \widehat{S}_1 (resp. \widehat{S}_{s+1}) is not middle at v . Hence by (2) and Lemma 6.3, the extended pseudo paths $\widehat{S}_1, \widehat{S}_{s+1}$ are I/O pseudo paths. Now for each $i = 1, 2, \dots, s$, we have that $(\widehat{S}_i, \widehat{S}_{i+1})$ is an I/O pair of type II. Thus $(\widehat{S}_1, \widehat{S}_2, \dots, \widehat{S}_s, \widehat{S}_{s+1})$ is an I/O sequence for (\widehat{S}, Δ) here Δ is a nice side-disk of \widehat{S} . Hence by Remark 5.4 the extended pseudo path $\widehat{S}_1 * \widehat{S}_2 * \dots * \widehat{S}_s * \widehat{S}_{s+1}$ is an I/O pseudo path and so is \widehat{S} . Similarly we can show that \widehat{T} is an I/O pseudo path. This completes the proof of Theorem 7.4. \square

8 Bridges

Let Γ be a chart. Let B be an admissible pseudo path of label m in Γ with the associated vertex sequence (v_1, v_2, \dots, v_s) and the associated edge sequence $(e_1, e_2, \dots, e_{s-1})$. The pseudo path B is called a *bridge* provided that

- (i) B contains at least two white vertices,
- (ii) the edge e_1 is not middle at v_1 and the edge e_{s-1} is not middle at v_s , and
- (iii) for each $i = 2, 3, \dots, s-1$, there exists a terminal edge of label m at v_i .

Since B is admissible, the vertices v_1 and v_s are white vertices. Thus (iii) implies that all the vertices in B are white vertices. Let γ_0 and γ_s be the end-arcs of B with $\gamma_0 \ni v_1$ and $\gamma_s \ni v_s$. Let Δ be a side-disk for which B is admissible. Let γ_0^* and γ_s^* be short arcs in edges of label m in Γ with $\gamma_0^* \cap \Delta = v_1$ and $\gamma_s^* \cap \Delta = v_s$. Then $B^* = \gamma_0^* \cup e_1 \cup e_2 \cup \dots \cup e_{s-1} \cup \gamma_s^*$ is a bridge called the *co-bridge* of B . It is clear that the bridge B is the co-bridge of B^* . If there exists a label k with $|m - k| = 1$ and $v_1, \dots, v_s \in \Gamma_m \cap \Gamma_k$, then the bridge B is called a *dichromatic bridge*.

Lemma 8.1 *Let Γ be a minimal chart. Let B be a dichromatic bridge of label m in Γ . Then one of the bridge B and the co-bridge B^* is an inward pseudo path, and the other is an outward pseudo path.*

PROOF. We use the notations in the definition of a bridge. Since the edge e_1 is not middle at v_1 by condition (ii) of a bridge, one of the two end-arcs γ_0 and γ_0^* is middle at v_1 by Remark 2.1(2). Since the co-bridge of B^* is B , without loss of generality we can assume that

- (1) the end-arc γ_0 is middle at v_1 .

Further, we can assume that

(2) the end-arc γ_0 is oriented inward at v_1 .

First, we prove the case $s = 2$. Then $B = \gamma_0 \cup e_1 \cup \gamma_2$ and $B^* = \gamma_0^* \cup e_1 \cup \gamma_2^*$ are monochromatic pseudo paths (see Fig. 16(a)). Let (γ'_1, γ'_2) and (γ''_1, γ''_2) be side-arc sequences of B and B^* respectively. Then by (1) and (2), we have

(3) the side-arc γ'_1 is oriented inward at v_1 , and the edge e_1 and the side-arc γ''_1 are oriented outward at v_1 .

Since the end-arc γ_0 is middle at v_1 by (1), and since $s = 2$, the bridge B is an I/O pseudo path by Lemma 6.3. Since the side-arc γ'_1 is oriented inward at v_1 by (3), the bridge B is an inward pseudo path. Thus the side-arc γ'_2 is oriented inward at v_2 . Now e_1 is oriented inward at v_2 by (3). Since e_1 is not middle at v_2 by condition (ii) of a bridge, the side-arc γ''_2 is oriented outward at v_2 . Since the side-arc γ''_1 is oriented outward at v_1 by (3), the co-bridge B^* is an outward pseudo path.

Suppose that $s \geq 3$. Now for each $i = 2, 3, \dots, s-1$ there exists a terminal edge \tilde{e}_i of label m at v_i by condition (iii) of a bridge. Since \tilde{e}_2 is a terminal edge at v_2 , by Remark 2.2(2) we have

(4) the edge \tilde{e}_2 is middle at v_2 .

If \tilde{e}_2 contains a side-arc of B , say $\tilde{\gamma}_2$, then $P = \gamma_0 \cup e_1 \cup \tilde{\gamma}_2$ is a monochromatic pseudo path (see Fig. 16(b)). But the end-arc γ_0 of P is middle at v_1 , and the end-arc $\tilde{\gamma}_2$ of P is middle at v_2 by (4). Further, the monochromatic pseudo path P contains only two white vertices v_1, v_2 . This contradicts Lemma 6.4(a). Hence \tilde{e}_2 contains a side-arc of B^* , say $\tilde{\gamma}_2^*$ (see Fig. 16(c)). By (4),

(5) the side-arc $\tilde{\gamma}_2^*$ is middle at v_2 .

Let γ_1, γ_2 be short arcs in the edges e_1, e_2 containing v_2 respectively. Let $B_1 = \gamma_0 \cup e_1 \cup \gamma_2, B_2 = \gamma_1 \cup e_2 \cup \dots \cup e_{s-1} \cup \gamma_s, B_1^* = \gamma_0^* \cup e_1 \cup \tilde{\gamma}_2^*, B_2^* = \tilde{\gamma}_2^* \cup e_2 \cup \dots \cup e_{s-1} \cup \gamma_s^*$. Since $\tilde{\gamma}_2^*$ is middle at v_2 by (5), the edges e_1 and e_2 are not middle at v_2 by Remark 2.1(2). Thus B_1, B_2, B_1^*, B_2^* are bridges.

Now B_1^*, B_2^* are co-bridges of B_1, B_2 respectively. Thus B_1 is inward and B_1^* is outward by the case $s = 2$. By induction on the number of edges of a bridge, we can show that B_2 and the co-bridge of B_2 are I/O pseudo paths. Since (B_1, B_2) is an I/O pair of type I, the bridge B is an I/O pseudo path by Lemma 5.2. Since B_1 is an inward pseudo path, so is B . Since (B_1^*, B_2^*) is an I/O pair of type II, the bridge B^* is an I/O pseudo path by Lemma 5.3. Since B_1^* is an outward pseudo path, so is B^* . This proves Lemma 8.1. \square

Let Γ be a chart. Let B be a pseudo path of label m in Γ with the associated vertex sequence (v_1, v_2, \dots, v_s) and the associated edge sequence $(e_1, e_2, \dots, e_{s-1})$. Let γ_0 and γ_s be the end-arcs of B with $\gamma_0 \ni v_1$ and $\gamma_s \ni v_s$. The pseudo path B is called a *pier* provided that

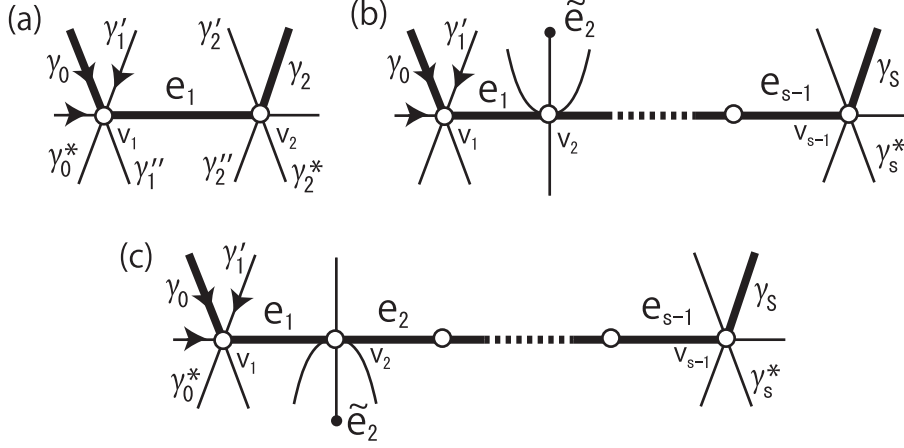


Figure 16: The thickened lines are bridges B .

- (i) v_1 is a white vertex,
- (ii) the edge e_1 is not middle at v_1 , and
- (iii) for each $i = 2, 3, \dots, s$, there exists a terminal edge of label m at v_i which does not contain the end-arc γ_s .

By (i) and (iii), all the vertices in B are white vertices. Let γ_0^* be a short arc in an edge of label m in Γ with $v_1 = \gamma_0^* \cap e_1 = \gamma_0^* \cap \gamma_0$. Then $B^* = \gamma_0^* \cup e_1 \cup e_2 \cup \dots \cup e_{s-1} \cup \gamma_s$ is a pier called the *co-pier* of B . It is clear that the pier B is the co-pier of B^* . There exist side-disks Δ and Δ^* of B and B^* respectively with $\Delta \cap \Delta^* = B \cap B^*$. The side-disk Δ (resp. Δ^*) is called a *nice side-disk* for B (resp. B^*). If there exists a label k with $|m - k| = 1$ and $v_1, \dots, v_s \in \Gamma_m \cap \Gamma_k$, then the pier B is called a *dichromatic pier*.

Corollary 8.2 *Let Γ be a minimal chart. Let B be a dichromatic pier of label m in Γ , and B^* the co-pier of B . Then one of the pier B and the co-pier B^* is an inward pseudo path with respect to a nice side-disk, and the other is an outward pseudo path with respect to a nice side-disk.*

PROOF. We use the notations in the definition of a pier. Let Δ and Δ^* be nice side-disks of B and B^* respectively. Our result is true for the case $s = 1$, because e_1 is not middle at v_1 . Suppose that $s \geq 2$. Let \tilde{e}_s be the terminal edge of label m at v_s not containing the end-arc γ_s . If $\tilde{e}_s \cap \text{Int} \Delta^* \neq \emptyset$, then B is an admissible pseudo path. If $\tilde{e}_s \cap \text{Int} \Delta \neq \emptyset$, then B^* is an admissible pseudo path. Since B and B^* are co-piers of each other, without loss of generality we can assume that B is an admissible pseudo path (see Fig. 17). Now the terminal edge \tilde{e}_s is middle at v_s by Remark 2.2(2). By Remark 2.1(2), the edge e_{s-1} is not middle at v_s . Thus B is a bridge. Let $\tilde{\gamma}_s$ be a short arc in the terminal edge \tilde{e}_s of label m with $v_s \in \tilde{\gamma}_s$. Then $B^\dagger = \gamma_0^* \cup e_1 \cup \dots \cup e_{s-1} \cup \tilde{\gamma}_s$ is a co-bridge of B . Thus B^\dagger and B are I/O

pseudo paths by Lemma 8.1. Without loss of generality we can assume that B is inward and B^\dagger is outward. Then a side-arc at v_s of B with respect to Δ is oriented inward at v_s . Hence \tilde{e}_s is oriented outward at v_s . Since the terminal edge \tilde{e}_s is middle at v_s , the side-arcs at v_s of B^* with respect to Δ^* is oriented outward at v_s . Thus the pier B^* is an outward pseudo path with respect to Δ^* . This proves Corollary 8.2. \square

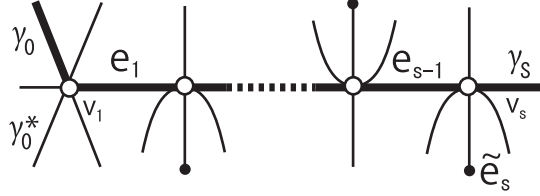


Figure 17: The thick line is a pier B .

9 Suspicious cycles

Let G be a graph. For each vertex v of G , we denote by $\deg_G v$ the degree of the vertex v in G .

Let Γ be a chart, and m a label of Γ . A tree T of Γ_m is called a *reducible tree* of label m provided that

- (i) the tree T contains at least two white vertices,
- (ii) for each white vertex v of T , we have $\deg_T v = 1$ or 3 , and
- (iii) there exists at most one white vertex v_0 in T with $\deg_T v_0 = 1$.

The white vertex v_0 with $\deg_T v_0 = 1$ is called *the special vertex* of the reducible tree.

Remark 9.1 For a chart Γ , any crossing is not considered as a vertex of Γ_m according to the definition of Γ_m , but an edge of Γ_m may contain a crossing. Thus a reducible tree may contain a crossing which is not a vertex of the tree.

Lemma 9.2 *Any minimal chart does not contain a reducible tree of any label.*

PROOF. Suppose that there exists a reducible tree T of label m in a minimal chart. Let T^* be the subtree obtained from the reducible tree T by taking out all the terminal edges. Since T^* is a tree containing at least two white vertices, there exist two white vertices each of whose degree in T^* is

1. Let w be one of them different from the special vertex. By condition (ii) for a reducible tree, we have $\deg_T w = 3$. Hence in the reducible tree T there exist two terminal edges of label m at w . By Remark 2.1(2), one of the two terminal edges is not middle at w . This contradicts Remark 2.2(2). \square

Let Γ be a chart. A tangle $(\Gamma \cap D, D)$ is said to be *admissible* provided that for any label k if an edge e of Γ_k intersects ∂D , then it is a normal edge and each connected component of $e \cap D$ contains a white vertex.

Remark 9.3 *If $(\Gamma \cap D, D)$ is an admissible tangle, then D contains all the terminal edges intersecting D . For, any terminal edge is neither a normal edge nor intersects ∂D .*

Let Γ be a chart. A tangle $(\Gamma \cap D, D)$ is said to be *two-color* if there exist two labels m, k with $|m - k| = 1$, and $\Gamma \cap D \subset \Gamma_m \cup \Gamma_k$.

Let C be a cycle of label m in a chart Γ . Then the cycle C is said to be *suspicious* provided that

- (i) the outside edges of Γ_m for C are terminal edges except one,
- (ii) the cycle contains a white vertex, and
- (iii) the cycle bounds a disk E with $\Gamma_m \cap E$ connected.

The following lemma is an easy consequence of the definition of a suspicious cycle.

Lemma 9.4 *In a minimal chart Γ , for any label m of Γ there does not exist a suspicious cycle of label m bounding a two-color disk.*

PROOF. Suppose that there exists a suspicious cycle C of label m bounding a two-color disk. By condition (i) of a suspicious cycle, we have $|\mathcal{O}(C) - \mathcal{O}_M(C)| \leq 1$. This contradicts Lemma 7.6(b). \square

Let X be a subset of a chart. A connected component G of X is called a *small component* of X if any finite complementary domain of G does not intersect X .

Let X be a subset of a chart. A cycle in X is said to be *maximal in X* if it is not contained in the disk bounded by another cycle in X .

Let Γ be a chart, and m a label of Γ . Let $(\Gamma \cap D, D)$ be an admissible tangle with $\Gamma_m \cap D \neq \emptyset$. Suppose that D contains neither hoop nor ring of label m . Let G be a small component of $\Gamma_m \cap D$, and E_1, E_2, \dots, E_d all the disks bounded by the maximal cycles in G . For each $i = 1, 2, \dots, d$, let

$$\widehat{E}_i = E_i \cup \{e \mid e \text{ is a terminal edge in } G \text{ intersecting } E_i\}.$$

Let P_1, P_2, \dots, P_p be all the closures of connected components of $G - \cup_{i=1}^d \widehat{E}_i$ not intersecting ∂D , and Q_1, Q_2, \dots, Q_q all the closures of connected components of $G - \cup_{i=1}^d \widehat{E}_i$ intersecting ∂D . Let

$$\begin{aligned}\mathcal{H} &= (\cup_{i=1}^d \widehat{E}_i) \cap ((\cup_{j=1}^p P_j) \cup (\cup_{k=1}^q Q_k)), & h &= |\mathcal{H}|, \\ s &= |(\cup_{i=1}^d \widehat{E}_i) \cap (\cup_{j=1}^p P_j)|, & t &= |(\cup_{i=1}^d \widehat{E}_i) \cap (\cup_{k=1}^q Q_k)|.\end{aligned}$$

For each $k = 0, 1, \dots, h$, let

x_k = the number of E_i 's containing exactly k points in \mathcal{H} ,

y_k = the number of P_j 's containing exactly k points in \mathcal{H} .

Then $(E_1, E_2, \dots, E_d; P_1, P_2, \dots, P_p; Q_1, Q_2, \dots, Q_q; \mathcal{H}, h, s, t; x_0, x_1, \dots, x_h, y_0, y_1, \dots, y_h)$ is called the *fundamental information* of G for the tangle $(\Gamma \cap D, D)$. This will be used in Lemma 9.5, Lemma 9.6, and the proof of Theorem 1.2.

Lemma 9.5 *Let Γ be a minimal chart, and m a label of Γ . Let $(\Gamma \cap D, D)$ be an admissible tangle with $\Gamma_m \cap D \neq \emptyset$. Suppose that D contains neither hoop nor ring of label m . Let G be a small component of $\Gamma_m \cap D$ with the fundamental information $(E_1, E_2, \dots, E_d; P_1, P_2, \dots, P_p; Q_1, Q_2, \dots, Q_q; \mathcal{H}, h, s, t; x_0, x_1, \dots, x_h, y_0, y_1, \dots, y_h)$ for the tangle $(\Gamma \cap D, D)$. Then we have the following:*

- (a) $y_0 = y_1 = 0$.
- (b) $2x_0 + x_1 = 2 - 2q + t + (x_3 + y_3) + 2(x_4 + y_4) + \dots + (h - 2)(x_h + y_h)$.
- (c) If $1 \leq x_0$ or $1 \leq x_1$, then D contains a suspicious cycle of label m .

PROOF. For Statement (a). Suppose $1 \leq y_0$ or $1 \leq y_1$. Then there exists a tree P_j ($1 \leq j \leq p$) containing at most one point of \mathcal{H} . Hence P_j is a reducible tree. This contradicts Lemma 9.2.

For Statement (b). For each $i = 1, 2, \dots, p$ and $j = 1, 2, \dots, q$ we have $P_i \cap Q_j = \emptyset$. Hence

- (1) $h = s + t$,
- (2) $x_0 + x_1 + \dots + x_h = d, y_0 + y_1 + \dots + y_h = p$, and
- (3) $1 \times x_1 + 2 \times x_2 + \dots + h \times x_h = h, 1 \times y_1 + 2 \times y_2 + \dots + h \times y_h = s = h - t$.

By adding the two equations in (3) we have

- (4) $(1 \times x_1 + 2 \times x_2 + \dots + h \times x_h) + (1 \times y_1 + 2 \times y_2 + \dots + h \times y_h) = 2 \times h - t$.

For each $i = 1, 2, \dots, d$, let

$$\widehat{E}_i = E_i \cup \{e \mid e \text{ is a terminal edge in } G \text{ intersecting } E_i\}.$$

Let $E_* = \cup_{i=1}^d \widehat{E}_i, P_* = \cup_{j=1}^p P_j, Q_* = \cup_{k=1}^q Q_k$ and $X = E_* \cup P_* \cup Q_*$. Then we have $X = G \cup (\cup_{i=1}^d E_i)$. Hence the set X is connected and any cycle in X bounds a disk in X , i.e. X is simply connected. Hence by Euler formula, we have $\chi(X) = 1$. On the other hand, considering (1) we have $\chi(X) = \chi(E_* \cup P_* \cup Q_*)$

$$= \chi(E_*) + \chi(P_*) + \chi(Q_*) - (\chi(E_* \cap P_*) + \chi(E_* \cap Q_*) + \chi(P_* \cap Q_*)) + \chi(E_* \cap P_* \cap Q_*)$$

$$= d + p + q - (s + t + 0) + 0 = d + p + q - h.$$

Hence

$$(5) \quad d + p + q - h = 1.$$

By doubling both sides of the equation (5), and eliminating d, p, h using (2) and (4), we have

$$2(x_0 + x_1 + \cdots + x_h) + 2(y_0 + y_1 + \cdots + y_h) + 2q - ((1 \times x_1 + 2 \times x_2 + \cdots + h \times x_h) + (1 \times y_1 + 2 \times y_2 + \cdots + h \times y_h) + t) = 2.$$

Since $y_0 = y_1 = 0$ by Lemma 9.5(a), we have

$$2x_0 + x_1 = 2 - 2q + t + (x_3 + y_3) + 2(x_4 + y_4) + \cdots + (h - 2)(x_h + y_h).$$

For Statement (c). Suppose $1 \leq x_0$ or $1 \leq x_1$. Then there exists a disk E_i ($1 \leq i \leq d$) intersecting \mathcal{H} by at most one point. Thus the outside edges of Γ_m for ∂E_i are terminal edges except one. Since G is a small component of $\Gamma_m \cap D$, the intersection $\Gamma_m \cap E_i$ is connected. Further there is no hoop nor ring in D . Thus the cycle ∂E_i contains a white vertex. Therefore ∂E_i is a suspicious cycle of label m . \square

Lemma 9.6 *Let Γ be a minimal chart, and m a label of Γ . Let $(\Gamma \cap D, D)$ be an admissible tangle with $\Gamma_m \cap D \neq \emptyset$ and $|\Gamma_m \cap \partial D| \leq 1$. If D contains neither hoop nor ring of label m , then D contains a suspicious cycle of label m .*

PROOF. Let G be a small component of $\Gamma_m \cap D$. If G contains a loop, then the loop is a suspicious cycle of label m . Hence we assume that

(1) G does not contain a loop.

Claim 1. G contains a white vertex.

Proof of Claim 1. If G intersects ∂D , then G contains the white vertex of an edge intersecting ∂D , because the tangle is admissible. Suppose that G does not intersect ∂D . Since D does not contain any free edges by Assumption 2, and since D contains neither hoop nor ring of label m by the assumption of our lemma, the small component G must contain a white vertex. Thus Claim 1 holds.

Claim 2. G contains a normal edge of Γ_m .

Proof of Claim 2. There exists a white vertex w in G by Claim 1. Suppose that G does not contain a normal edge of Γ_m . Since G does not contain a loop by (1), and since $|\Gamma_m \cap \partial D| \leq 1$, we have G contains at least two terminal edges of label m at w . By Remark 2.1(2), one of the two terminal edges is not middle at w . This contradicts Remark 2.2(2). Thus G contains a normal edge of Γ_m . Thus Claim 2 holds.

Claim 3. G contains a cycle.

Proof of Claim 3. Suppose that G does not contain a cycle. Since G contains a normal edge by Claim 2, the component G contains at least two white vertices. If G does not intersect ∂D , then G is a reducible tree without

the special vertex. This contradicts Lemma 9.2. Suppose that G intersects ∂D . Again $|\Gamma_m \cap \partial D| \leq 1$ implies that there exist an edge e^* of Γ_m and its white vertex v^* with $e^* \cap \partial D \neq \emptyset$, $(G - e^*) \cap \partial D = \emptyset$, and $G \not\ni v^*$. Let $T = G \cup e^*$. Then for any white vertex v of G , we have $\deg_T v = 3$. Hence T is a reducible tree with the special vertex v^* . This contradicts Lemma 9.2. Thus Claim 3 holds.

Let $(E_1, E_2, \dots, E_d; P_1, P_2, \dots, P_p; Q_1, Q_2, \dots, Q_q; \mathcal{H}, h, s, t; x_0, x_1, \dots, x_h, y_0, y_1, \dots, y_h)$ be the fundamental information of the small component G for the tangle $(\Gamma \cap D, D)$. By Lemma 9.5(b), we have

$$(2) \quad 2x_0 + x_1 = 2 - 2q + t + (x_3 + y_3) + 2(x_4 + y_4) + \dots + (h - 2)(x_h + y_h).$$

Now $|\Gamma_m \cap \partial D| \leq 1$ implies that $q = 0$ or 1 . If $q = 0$ then $2 - 2q + t \geq 2$. Hence $2x_0 + x_1 \geq 2$ by (2). Thus $x_0 \geq 1$ or $x_1 \geq 1$. Hence there exists a suspicious cycle of label m by Lemma 9.5(c). Suppose $q = 1$. By Claim 3, we have $d \geq 1$. Since $G \cup (\cup_{i=1}^d E_i)$ is connected, we have $t \geq 1$. Thus $2 - 2q + t \geq 1$. Hence $2x_0 + x_1 \geq 1$ by (2). Thus $x_0 \geq 1$ or $x_1 \geq 1$. Therefore there exists a suspicious cycle of label m by Lemma 9.5(c). This proves Lemma 9.6. \square

Lemma 9.7 *Let Γ be a minimal chart. Let $(\Gamma \cap D, D)$ be a two-color admissible tangle. Then D contains neither hoop nor ring.*

PROOF. Since $(\Gamma \cap D, D)$ is a two-color tangle, the disk D does not contain any crossing. Hence D does not contain any ring.

Suppose that D contains a hoop. Since $(\Gamma \cap D, D)$ is a two-color tangle, there exist two labels m, k with $\Gamma \cap D \subset \Gamma_m \cup \Gamma_k$ and $|m - k| = 1$. Let H be an innermost hoop in D , and E the disk bounded by the hoop H . Then $\text{Int} E$ contains neither hoop nor ring. Further, by Assumption 2, the hoop H is not simple. Hence the disk E contains a white vertex w . Since $\Gamma \cap D \subset \Gamma_m \cup \Gamma_k$ with $|m - k| = 1$, the vertex w is in Γ_m . Hence there exists a disk D' with $(\Gamma \cap E) - H \subset \text{Int} D' \subset D' \subset \text{Int} E$. Then $(\Gamma \cap D', D')$ is a two-color admissible tangle with $\Gamma_m \cap D' \neq \emptyset$ and $|\Gamma_m \cap \partial D'| = 0$. Since $\text{Int} E$ does not contain a hoop nor a ring, neither does D' . Thus by Lemma 9.6, the disk D' contains a suspicious cycle C of label m . Since $(\Gamma \cap D, D)$ is a two-color tangle, the suspicious cycle C bounds a two-color disk. This contradicts Lemma 9.4. \square

Lemma 9.8 *Let Γ be a minimal chart, and m a label of Γ . Then there does not exist a two-color admissible tangle $(\Gamma \cap D, D)$ with $\Gamma_m \cap D \neq \emptyset$ and $|\Gamma_m \cap \partial D| \leq 1$.*

PROOF. Suppose that there exists such a two-color admissible tangle $(\Gamma \cap D, D)$. By Lemma 9.7 the disk D does not contain a hoop nor a ring of label m . Since $\Gamma_m \cap D \neq \emptyset$ and $|\Gamma_m \cap \partial D| \leq 1$, by Lemma 9.6 the disk D contains a suspicious cycle of label m . Since $(\Gamma \cap D, D)$ is a two-color tangle,

the suspicious cycle bounds a two-color disk. This contradicts Lemma 9.4. \square

10 IO-tangles

Let Γ be a chart, and $(\Gamma \cap D, D)$ an admissible tangle. Let m be a label of Γ with $\Gamma_m \cap D \neq \emptyset$, and G a connected component of $\Gamma_m \cap D$. Set

$$X = \cup \{E \mid E \text{ is a disk bounded by a cycle in } G\}, \text{ and}$$

$$Y = \cup \{e \mid e \text{ is a terminal edge intersecting } G\}.$$

Then by Remark 9.3, $Y \subset \text{Int}D$. Let $G^* = G \cup X \cup Y$. Since any terminal edge does not contain a crossing by Remark 2.2(1), G^* is simply connected. Let D^* be a regular neighbourhood of G^* in D . Then D^* is a disk. Thus $(\Gamma \cap D^*, D^*)$ is a tangle. The tangle $(\Gamma \cap D^*, D^*)$ is called a *tangle induced from G* .

Remark 10.1 (1) Since D^* is a regular neighbourhood of $G^* = G \cup X \cup Y$ in D , and $(X \cup Y) \cap \partial D = \emptyset$, we have

$$|\Gamma_m \cap \partial D^*| = |G \cap \partial D^*| = |G \cap \partial D|.$$

- (2) If an edge intersects ∂D^* , then the edge must intersect G , because D^* is a regular neighborhood of G^* .
- (3) If $(\Gamma \cap D, D)$ is an admissible tangle without crossings, then $(\Gamma \cap D^*, D^*)$ is admissible. For, let e be an edge of Γ_k with $e \cap \partial D^* \neq \emptyset$, and γ a connected component of $e \cap D^*$. Then γ must intersect G , because D^* is a regular neighborhood of G^* . Since there is no crossing in D , the connected component γ contains a white vertex in $G \subset D^*$. Since $Y \subset G^*$, any terminal edge does not intersect ∂D^* . Therefore $(\Gamma \cap D^*, D^*)$ is admissible.

Let C be a cycle of label m in a chart Γ , and E the disk bounded by C . Let

$$E^* = E \cup \{e \mid e \text{ is a terminal edge intersecting } E\}.$$

Let D^* be a regular neighbourhood of E^* . Then $(\Gamma \cap D^*, D^*)$ is a tangle. The tangle $(\Gamma \cap D^*, D^*)$ is called a *tangle induced from the cycle C* .

Remark 10.2 (1) $\text{Int}D^*$ contains all terminal edges intersecting C .

- (2) If E does not contain any crossing, then D^* does not contain any crossing.
- (3) If an edge intersects ∂D^* , then it must intersect C , because D^* is a regular neighborhood of E^* .

- (4) If the cycle C contains at most one crossing, then $(\Gamma \cap D^*, D^*)$ is admissible. For, let e be an edge of Γ_k with $e \cap \partial D^* \neq \emptyset$. Suppose that e does not contain a white vertex in D^* . Then e must intersect C at a crossing by (3). Hence e is not a terminal edge by Remark 2.2(1). Since C contains at most one crossing, the edge e does not intersect C twice. Hence e is neither a loop nor a ring. Thus e is a normal edge.

Lemma 10.3 *Let Γ be a minimal chart. Let E be a two-color disk bounded by a cycle C of label m in Γ . Then $\Gamma_m \cap E$ is connected.*

PROOF. Let $(\Gamma \cap D^*, D^*)$ be a tangle induced from the cycle C . Then $E \subset D^*$. Since E does not contain any crossing, neither does the cycle C . Thus $(\Gamma \cap D^*, D^*)$ is admissible by Remark 10.2(4).

Suppose that $\Gamma_m \cap E$ is not connected. Let G be a small component of $\Gamma_m \cap E$ with $G \cap \partial E = \emptyset$. Since $E \subset D^*$, we have $G \cap E = G \cap D^*$. Let $(\Gamma \cap D, D)$ be a tangle induced from G . By Remark 10.1(1) we have $|\Gamma_m \cap \partial D| = |G \cap \partial D^*| = 0$. Since E does not contain any crossing, the disk D^* does not contain any crossing by Remark 10.2(2). Since $(\Gamma \cap D^*, D^*)$ is an admissible tangle without crossing, by Remark 10.1(3) the tangle $(\Gamma \cap D, D)$ is admissible. Since E is a two-color disk, the tangle $(\Gamma \cap D, D)$ is a two-color tangle. Hence the tangle $(\Gamma \cap D, D)$ is a two-color admissible tangle with $\Gamma_m \cap D \neq \emptyset$ and $|\Gamma_m \cap \partial D| = 0$. This contradicts Lemma 9.8. \square

Lemma 10.4 *Let Γ be a minimal chart, and $(\Gamma \cap D, D)$ a two-color admissible tangle. Let m be a label of Γ with $\Gamma_m \cap D \neq \emptyset$. If there exist two distinct pseudo paths T_α, T_β in $\Gamma_m \cap D$ satisfying the three conditions (i), (ii) and (iii) in the definition of an IO-tangle, then the tangle is an IO-tangle.*

PROOF. We use the notations in the definition of an IO-tangle. For each $i = 1, 2, \dots, s$, let (see Fig. 18)

$$\begin{aligned} v_i &= E_i \cap L_{i-1}, & w_i &= E_i \cap L_i, \\ \gamma_i &= \text{an arc in the edge in } E_i \cap \Delta_\alpha \text{ with } v_i \in \gamma_i, \\ \delta_i &= \text{an arc in the edge in } E_i \cap \Delta_\alpha \text{ with } w_i \in \delta_i, \\ \phi_i &= \text{an arc in the edge in } L_{i-1} \text{ with } v_i \in \phi_i, \\ \psi_i &= \text{an arc in the edge in } L_i \text{ with } w_i \in \psi_i, \\ T_i &= E_i \cap \Delta_\alpha, & \widehat{T}_i &= \phi_i \cup T_i \cup \psi_i. \end{aligned}$$

For each $i = 0, 1, \dots, s$, let $\widehat{L}_i = \delta_i \cup L_i \cup \gamma_{i+1}$, here $\delta_0 = \emptyset, \gamma_{s+1} = \emptyset$.

Since $(\Gamma \cap D, D)$ is a two-color tangle, for each $i = 1, 2, \dots, s$ the disk E_i is a two-color disk. By Lemma 10.3,

- (1) for each $i = 1, 2, \dots, s$, the intersection $\Gamma_m \cap E_i$ is connected.

Claim 1. The pseudo paths $\widehat{L}_0, \widehat{L}_1, \dots, \widehat{L}_s$ are I/O pseudo paths with respect to Δ_α .

Proof of Claim 1. For each $i = 1, 2, \dots, s$, by condition (ii) and condition (iii)(c) of an IO-tangle, we have $|\mathcal{O}(\partial E_i) - \mathcal{O}_M(\partial E_i)| \leq 2$. Hence by (1) and Lemma 7.6(b), we have $|\mathcal{O}(\partial E_i) - \mathcal{O}_M(\partial E_i)| = 2$. Hence

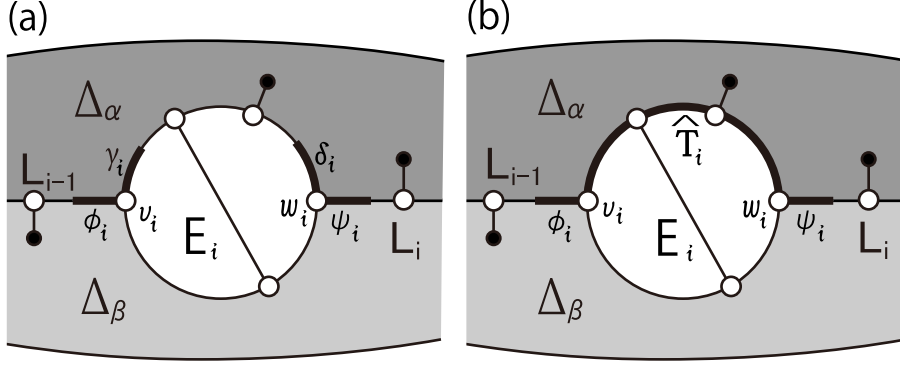


Figure 18: (a) The thick lines are arcs $\gamma_i, \delta_i, \phi_i$ and ψ_i . (b) The thick line is a pseudo path \widehat{T}_i .

$$(2) \{v_i, w_i\} = \mathcal{O}(\partial E_i) - \mathcal{O}_M(\partial E_i).$$

Thus the arc ϕ_i is not middle at v_i , nor the arc ψ_i is not middle at w_i . Hence by condition (iii)(c) of an IO-tangle, for each $i = 1, 2, \dots, s-1$ the pseudo path \widehat{L}_i is a dichromatic bridge. Further \widehat{L}_0 and \widehat{L}_s are dichromatic piers because ϕ_1 and ψ_s are not middle at v_1 and w_s respectively. Thus Claim 1 follows from Lemma 8.1 and Corollary 8.2. Hence Claim 1 holds.

By (2), Theorem 7.3(b) implies that

- (3) for each $i = 1, 2, \dots, s$, the extended pseudo path \widehat{T}_i of the path T_i is an I/O pseudo path with respect to Δ_α .

Claim 2. T_α, T_β are I/O pseudo paths with respect to $\Delta_\alpha, \Delta_\beta$ respectively.

Proof of Claim 2. Let $i \in \{1, 2, \dots, s\}$. Since $\widehat{L}_{i-1} \cap \widehat{T}_i$ contains the white vertex v_i , and since $\widehat{T}_i \cap \widehat{L}_i$ contains the white vertex w_i , the pairs $(\widehat{L}_{i-1}, \widehat{T}_i)$ and $(\widehat{T}_i, \widehat{L}_i)$ are I/O pairs of type I by Claim 1 and (3). Thus $(\widehat{L}_0, \widehat{T}_1, \widehat{L}_1, \widehat{T}_2, \dots, \widehat{L}_{s-1}, \widehat{T}_s, \widehat{L}_s)$ is an I/O sequence for (T_α, D_α) . Hence by Remark 5.4, $T_\alpha = \widehat{L}_0 * \widehat{T}_1 * \widehat{L}_1 * \widehat{T}_2 * \dots * \widehat{L}_{s-1} * \widehat{T}_s * \widehat{L}_s$ is an I/O pseudo path with respect to Δ_α .

Similarly T_β is an I/O pseudo path with respect to Δ_β . Thus Claim 2 holds.

Let γ'_α be a side-arc of T_α at v_1 with respect to Δ_α , and γ'_β a side-arc of T_β at v_1 with respect to Δ_β . Since ϕ_1 is not middle at v_1 by (2), one of γ'_α and γ'_β is oriented inward at v_1 and the other is oriented outward at v_1 . Since γ'_α is a side-arc of T_α with respect to Δ_α and since γ'_β is a side-arc of T_β with respect to Δ_β , one of T_α and T_β is inward and the other is outward with respect to $\Delta_\alpha, \Delta_\beta$. This proves Lemma 10.4. \square

Proof of Theorem 1.2. The tangle $(\Gamma \cap D, D)$ is a two-color admissible tangle with $|\Gamma_m \cap \partial D| = 2$. By Lemma 9.7, the disk D contains neither hoop nor ring of label m .

Claim 1. $\Gamma_m \cap D$ is connected.

Proof of Claim 1. If $\Gamma_m \cap D$ is not connected, then there exists a small component G of $\Gamma_m \cap D$ with $|G \cap \partial D| \leq 1$. Let $(\Gamma \cap D', D')$ be a tangle induced from G . Since $(\Gamma \cap D, D)$ is a two-color tangle, we have D does not contain any crossing and $(\Gamma \cap D', D')$ is a two-color tangle. Since $(\Gamma \cap D, D)$ is admissible without crossings, the tangle $(\Gamma \cap D', D')$ is admissible by Remark 10.1(3). By Remark 10.1(1) we have $|\Gamma_m \cap \partial D'| = |G \cap \partial D| \leq 1$. Hence $(\Gamma \cap D', D')$ is a two-color admissible tangle with $|\Gamma_m \cap \partial D'| \leq 1$. This contradicts Lemma 9.8. Hence Claim 1 holds.

Since $\Gamma_m \cap D$ is connected, let $(E_1, E_2, \dots, E_d; P_1, P_2, \dots, P_p; Q_1, Q_2, \dots, Q_q; \mathcal{H}, h, s, t; x_0, x_1, \dots, x_h, y_0, y_1, \dots, y_h)$ be the fundamental information of the small component $\Gamma_m \cap D$ for the tangle $(\Gamma \cap D, D)$.

Claim 2. $x_0 = x_1 = 0$.

Proof of Claim 2. The tangle $(\Gamma \cap D, D)$ is a two-color tangle by condition (a) in Theorem 1.2. If $1 \leq x_0$ or $1 \leq x_1$, then there exists a suspicious cycle in D by Lemma 9.5(c). Since the tangle $(\Gamma \cap D, D)$ is a two-color tangle, the suspicious cycle bounds a two-color disk. This contradicts Lemma 9.4. Hence $x_0 = x_1 = 0$. Therefore Claim 2 holds.

Now $|\Gamma_m \cap \partial D| = 2$ by condition (b) in Theorem 1.2. Hence $q = 1$ or 2 . Also we have $d \geq 1$ by condition (c) in Theorem 1.2. Since $\Gamma_m \cap D$ is connected, we have $t \geq 1$.

Suppose $q = 1$. Putting the values $x_0 = x_1 = 0$ and $q = 1$ into the equation in Lemma 9.5(b), we have

$$(1) \quad 0 = 2 - 2 \times 1 + t + (x_3 + y_3) + 2(x_4 + y_4) + \dots + (h - 2)(x_h + y_h).$$

Hence $t = 0$. This contradicts the fact $t \geq 1$. Thus $q = 2$. Since $\Gamma_m \cap D$ is connected, we have $t \geq 2$. Again putting the values $x_0 = x_1 = 0$ and $q = 2$ into the equation in Lemma 9.5(b), we have

$$(2) \quad 0 = 2 - 2 \times 2 + t + (x_3 + y_3) + 2(x_4 + y_4) + \dots + (h - 2)(x_h + y_h).$$

Since $t \geq 2$, we have $x_3 = x_4 = \dots = x_h = y_3 = \dots = y_h = 0$ and $t = 2$. Hence by Claim 2, Lemma 9.5(a) and the definition of the fundamental information of $\Gamma_m \cap D$, we have

- (3) each of E_1, E_2, \dots, E_d intersects exactly two of $P_1, P_2, \dots, P_p, Q_1, Q_2$ by exactly one point with each of them,
- (4) each of P_1, P_2, \dots, P_p intersects exactly two of E_1, E_2, \dots, E_d by exactly one point with each of them, and
- (5) each of Q_1, Q_2 intersects exactly one of E_1, E_2, \dots, E_d by exactly one point.

Hence we have $d = p + 1$. Thus by renumbering E_1, E_2, \dots, E_d and P_1, P_2, \dots, P_{d-1} , and by setting $P_0 = Q_1, P_d = Q_2$ we can assume that

$$(6) \quad \text{for each } i = 1, 2, \dots, d \text{ and } j = 0, 1, \dots, d, \\ E_i \cap P_j = \begin{cases} \text{one point} & \text{if } j = i - 1 \text{ or } i, \\ \emptyset & \text{otherwise.} \end{cases}$$

Set $w_0 = P_0 \cap \partial D$, and $v_{d+1} = P_d \cap \partial D$. For each $i = 1, 2, \dots, d$ and $j = 0, 1, \dots, d$, let

$$v_i = E_i \cap P_{i-1}, \quad w_i = E_i \cap P_i,$$

L_j = the simple arc in P_j connecting w_j and v_{j+1} .

Let $Y = (\cup_{i=1}^d E_i) \cup (\cup_{j=0}^d L_j)$ and $X = D - Y$.

Claim 3. All the edges in $Cl(X \cap \Gamma_m)$ are terminal edges.

Proof of Claim 3. If not, there exists a connected component T of $Cl(X \cap \Gamma_m)$ such that T is a tree containing at least two white vertices with $|T \cap Y| = 1$. Thus T is a reducible tree with the special vertex $T \cap Y$. This contradicts Lemma 9.2. Hence Claim 3 holds.

Further Y is simply connected and $Y \cap \partial D = \{w_0, v_{d+1}\}$. Thus

(7) the set $X = D - Y$ consists of two connected components.

Let $\Delta_\alpha, \Delta_\beta$ be the closures of the connected components. Then

(8) $\Delta_\alpha, \Delta_\beta$ are disks.

Set $T_\alpha = Cl(\partial\Delta_\alpha - \partial D)$, and $T_\beta = Cl(\partial\Delta_\beta - \partial D)$. Then

(9) T_α, T_β are pseudo paths connecting the two points w_0 and v_{d+1} ,

(10) $T_\alpha \cap \partial D = T_\beta \cap \partial D = \{w_0, v_{d+1}\} = \partial T_\alpha = \partial T_\beta$, and

(11) the closure of $(T_\alpha \cup T_\beta) - (T_\alpha \cap T_\beta)$ bounds mutually disjoint disks E_1, E_2, \dots, E_d .

Since $X = D - Y = D - (\cup_{i=1}^d E_i \cup T_\alpha \cup T_\beta)$, the pseudo paths T_α, T_β are desired pseudo paths in $\Gamma_m \cap D$ satisfying conditions (i), (ii) and (iii) for an IO-tangle. Therefore Theorem follows from Lemma 10.4. \square

11 NS-tangles

Let Γ be a chart, and D a disk. We define

$\omega(D)$ = the number of white vertices in $\Gamma \cap D$,

$x(D)$ = the number of crossings in $\Gamma \cap D$.

For a tangle $(\Gamma \cap D, D)$ in a chart Γ , let $\tau(D) = (\omega(D), x(D))$. We call $\tau(D)$ the τ -complexity of the tangle.

Let Γ be a chart. An NS-tangle $(\Gamma \cap D, D)$ of label m is said to be *minimal* if its τ -complexity of the tangle is minimal amongst the NS-tangles of all the labels with respect to the lexicographical order of the pair of integers.

Lemma 11.1 *Let Γ be a minimal chart. Let $(\Gamma \cap D, D)$ be a minimal NS-tangle of label m . Then D does not contain any ring.*

PROOF. Suppose that D contains a ring C of label k . Then C bounds a disk E . Let A be a regular neighbourhood of ∂E , and $D' = Cl(E - A)$. By condition (iii) of an NS-tangle, we have

- (1) for each label $i \neq k$, the ring $C = \partial E$ intersects Γ_i at most one point, and so does $\partial D'$.

By Remark 2.2(3), the disk E contains a white vertex. Thus we have

- (2) the disk D' contains a white vertex.

By condition (iii) of an NS-tangle, for each label i the intersection $\Gamma_i \cap D$ contains at most one crossing. Since $D' \subset D$, we have

- (3) for each label i the intersection $\Gamma_i \cap D'$ contains at most one crossing. Hence $(\Gamma \cap D', D')$ is an NS-tangle of label k . Since C is a ring with $C \cap D' = \emptyset$, we have $\omega(D') \leq \omega(D)$, and $x(D') < x(D)$. Hence we have $\tau(D') < \tau(D)$. This contradicts the fact that the NS-tangle $(\Gamma \cap D, D)$ is minimal. \square

Lemma 11.2 *If there exists an NS-tangle in a minimal chart Γ , then there exists a minimal NS-tangle $(\Gamma \cap D, D)$ such that D does not contain any hoop.*

PROOF. Let $(\Gamma \cap D^*, D^*)$ be a minimal NS-tangle of Γ . Suppose that D^* contains a hoop. Let C be an innermost hoop in D^* . Let E be the disk bounded by the hoop C , and A a regular neighbourhood of C . Set $D = Cl(E - A)$. Then $\Gamma \cap \partial D = \emptyset$. Since C is an innermost hoop in D^* , the disk D does not contain any hoop. Since C is not simple by Assumption 2, the disk E contains a white vertex and so does D . Thus $(\Gamma \cap D, D)$ is an NS-tangle. Since $(\Gamma \cap D^*, D^*)$ is a minimal NS-tangle, so is $(\Gamma \cap D, D)$. Hence the tangle $(\Gamma \cap D, D)$ is a desired tangle. \square

For a subset X of a chart Γ , we define

$$\alpha(X) = \min\{i \mid \Gamma_i \cap X \neq \emptyset\}, \beta(X) = \max\{i \mid \Gamma_i \cap X \neq \emptyset\}.$$

Proof of Theorem 1.3. Suppose that there exists an NS-tangle in a minimal chart Γ . By Lemma 11.1 and Lemma 11.2, there exists a minimal NS-tangle $(\Gamma \cap D, D)$ of label m such that

- (1) D contains neither hoop nor ring.

Let $\alpha = \alpha(\Gamma \cap D)$, $\beta = \beta(\Gamma \cap D)$. Then $\alpha \neq m$ or $\beta \neq m$. Without loss of generality we can assume that $\alpha \neq m$. By condition (i) of an NS-tangle, we have $|\Gamma_\alpha \cap \partial D| \leq 1$. Thus by Lemma 9.6, in the tangle $(\Gamma \cap D, D)$ there exists a suspicious cycle C^* of label α in D . Let $(\Gamma \cap D^*, D^*)$ be a tangle induced from C^* . Since C^* contains at most one crossing by condition (iii) of an NS-tangle, $(\Gamma \cap D^*, D^*)$ is admissible by Remark 10.2(4).

Claim 1. The admissible tangle $(\Gamma \cap D^*, D^*)$ is a minimal NS-tangle of label $\alpha + 1$.

Proof of Claim 1. Since C^* is a suspicious cycle, we have

- (2) the cycle C^* contains a white vertex.

Let i be a label with $i \geq \alpha + 2$. Since $C^* \subset D$, the intersection $\Gamma_i \cap C^*$ consists of at most one crossing by condition (iii) of an NS-tangle. If an edge

of Γ_i intersects ∂D^* , then it must intersect the cycle C^* by Remark 10.2(3). Hence there exists at most one edge of Γ_i intersecting ∂D^* . Thus

(3) for any label $i \geq \alpha + 2$ the intersection $\Gamma_i \cap \partial D^*$ is at most one point. Further, since C^* is a suspicious cycle of label α ,

(4) the intersection $\Gamma_\alpha \cap \partial D^*$ is at most one point by Remark 10.2(1) and condition (i) of a suspicious cycle. Since α is the lowest label in D ,

(5) for any label $j < \alpha$, we have $\Gamma_j \cap \partial D^* = \emptyset$.

Since $(\Gamma \cap D, D)$ is an NS-tangle, for each label i , $\Gamma_i \cap D$ contains at most one crossing, and since $D^* \subset D$, we have

(6) for each label i , $\Gamma_i \cap D^*$ contains at most one crossing.

The tangle $(\Gamma \cap D^*, D^*)$ is an NS-tangle of label $\alpha + 1$ in D by (2), (3), (4), (5) and (6). Since the NS-tangle $(\Gamma \cap D, D)$ is minimal, so is the NS-tangle $(\Gamma \cap D^*, D^*)$. Hence Claim 1 holds.

Claim 2. The tangle $(\Gamma \cap D^*, D^*)$ is a two-color tangle.

Proof of Claim 2. Suppose that there exists an integer $r \neq \alpha, \alpha + 1$ with $\Gamma_r \cap D^* \neq \emptyset$. Then $r > \alpha + 1$, because α is the lowest label in D . Let $\beta^* = \beta(\Gamma \cap D^*)$. Then $\beta^* \geq r > \alpha + 1$. Thus $\Gamma_{\beta^*} \cap \partial D^*$ consists of at most one point by (3). By Assumption 2, the disk D does not contain any free edges. Thus by (1), we have

(7) D^* contains neither hoop, ring, nor free edge.

Hence $\Gamma_{\beta^*} \cap D^*$ contains a white vertex. Thus by (7), Lemma 9.6 assures that in the admissible tangle $(\Gamma \cap D^*, D^*)$ with $|\Gamma_{\beta^*} \cap D^*| \leq 1$, there exists a suspicious cycle C^{**} of label β^* in D^* . Let $(\Gamma \cap D^{**}, D^{**})$ be a tangle induced from C^{**} . Since $\beta^* \neq \alpha, \alpha + 1$, the cycle C^{**} is contained in the interior of the disk bounded by C^* . Hence (2) implies that

(8) $\omega(D^{**}) < \omega(D^*)$.

Now we can show that the tangle $(\Gamma \cap D^{**}, D^{**})$ is an NS-tangle of label $\beta^* - 1$ by a similar way as the one used to show Claim 1. Now by (8) we have $\tau(D^{**}) < \tau(D^*)$. This contradicts the fact that the NS-tangle $(\Gamma \cap D^*, D^*)$ is minimal. Hence Claim 2 holds.

Therefore the suspicious cycle C^* bounds a two-color disk. This contradicts Lemma 9.4. This proves Theorem 1.3. \square

12 Appendix

Lemma 12.1 ([12, Lemma 4.1]) [**Boundary Condition Lemma**] *Let $(\Gamma \cap D, D)$ be an admissible tangle in a minimal chart Γ such that D does not contain any crossing. Let $a = \alpha(\Gamma \cap \partial D)$ and $b = \beta(\Gamma \cap \partial D)$. Then $\Gamma_i \cap D = \emptyset$ except for $a \leq i \leq b$.*

PROOF. We claim that D does not contain any hoop. If D contains a hoop, say C' , then C' is not a simple hoop by Assumption 2. Hence C' bounds a

disk E' containing a white vertex. Let D' be a regular neighborhood of E' . Then $\Gamma \cap \partial D' = \emptyset$. Since D does not contain any crossing, neither does D' . Thus $(\Gamma \cap D', D')$ is an NS-tangle. This contradicts Theorem 1.3. Hence

(1) D does not contain any hoop.

Let i be a label with $\Gamma_i \cap D \neq \emptyset$. Suppose that $i < a$ or $b < i$. Without loss of generality we can assume $i < a$. Let $\alpha = \alpha(\Gamma \cap D)$. Since $\alpha \leq i < a = \alpha(\Gamma \cap \partial D)$, we have $\Gamma_\alpha \cap \partial D = \emptyset$. Let G be a small component of $\Gamma_\alpha \cap D$. Then $G \cap \partial D = \emptyset$. Let $(\Gamma \cap D^*, D^*)$ be a tangle induced from G . Then $D^* \subset D$. Since D does not contain any crossing by assumption, the tangle $(\Gamma \cap D^*, D^*)$ is admissible by Remark 10.1(3).

Since $G \cap \partial D = \emptyset$, by Remark 10.1(1), we have $|\Gamma_\alpha \cap \partial D^*| = |G \cap \partial D| = 0$. Hence

(2) $\Gamma_\alpha \cap \partial D^* = \emptyset$.

Since α is the lowest label in D ,

(3) for each label $j < \alpha$, we have $\Gamma_j \cap \partial D^* = \emptyset$.

Since D does not contain any crossing by assumption, for each label $j > \alpha + 1$ we have $G \cap \Gamma_j = \emptyset$. Hence by Remark 10.1(2)

(4) for each label $j > \alpha + 1$, we have $\Gamma_j \cap \partial D^* = \emptyset$.

Since $D^* \subset D$, by assumption

(5) D^* does not contain any crossing.

By (1), the disk D^* does not contain any hoop. By Assumption 2, the disk D^* does not contain any free edge. By (5), the disk D^* does not contain any ring. Hence D^* contains at least one white vertex. Thus by (2), (3), (4) and (5), the tangle $(\Gamma \cap D^*, D^*)$ is an NS-tangle of label $\alpha + 1$. This contradicts Theorem 1.3. Therefore $a \leq i \leq b$. \square

Lemma 12.2 *Let Γ be a minimal chart. Let C be a cycle of label m bounding a disk E without crossings. Then we have the following:*

- (a) E is a three-color disk.
- (b) *If there exists a label k with $|m - k| = 1$ such that all the white vertices in C are in $\Gamma_m \cap \Gamma_k$, then E is a two-color disk.*

PROOF. Let $(\Gamma \cap D, D)$ be a tangle induced from the cycle C . Then $E \subset D$. Since C does not contain any crossing, the tangle $(\Gamma \cap D, D)$ is admissible by Remark 10.2(4). Since ∂E is a cycle of label m , we have $m - 1 \leq \alpha(\Gamma \cap \partial D)$ and $\beta(\Gamma \cap \partial D) \leq m + 1$. By Boundary Condition Lemma (Lemma 12.1), we have $m - 1 \leq \alpha(\Gamma \cap D)$ and $\beta(\Gamma \cap D) \leq m + 1$. Thus $\Gamma \cap D \subset \Gamma_{m-1} \cup \Gamma_m \cup \Gamma_{m+1}$. Since $E \subset D$, the disk E is a three-color disk.

Similarly we can show Statement (b). \square

Let e', e, e'' be three consecutive edges containing a white vertex w where e lies between e' and e'' . The edges e' and e'' are called (e, w) -edges.

Proof of Lemma 3.1. Suppose that there exists a non-admissible consecutive triplet (e_1, e_2, e_3) in a minimal chart. Then e_1 is a terminal edge and the labels of e_1 and e_3 are same. Let $\partial e_2 = \{w_1, w_2\}$, $w_1 \in e_1$, and $w_2 \in e_3$ (possibly $w_1 = w_2$). Suppose that $\partial e_3 = \{w_2, w_3\}$ (possibly $w_2 = w_3$).

To make argument simple we assume that the three points w_1, w_2 and w_3 are mutually different. The vertex w_3 may not be a white vertex. Without loss of generality, we can assume that the edge e_1 is oriented outward at the white vertex w_1 . In a minimal chart, by Remark 2.2(2) any terminal edge must contain a middle arc at its white vertex. Hence e_1 contains a middle arc at w_1 . Thus the edge e_2 is oriented outward at the white vertex w_1 , too.

The edge e_3 is oriented inward at the white vertex w_2 . For, if the edge e_3 is oriented outward at w_2 , then the edge e_3 contains a non-middle arc at w_2 . By applying a C-I-M2 move between e_1 and e_3 we can get a new terminal edge which contains a non-middle arc at the white vertex w_2 . Hence we can eliminate the white vertex w_2 by a C-III move. This contradicts that the chart is minimal. Therefore the edge e_3 is oriented inward at the white vertex w_2 (see Fig. 19(a)).

Let e_4 be the (e_2, w_1) -edge different from the edge e_1 . Since e_1 contains a middle arc and oriented outward at w_1 , the edge e_4 is oriented inward at the white vertex w_1 . Let e_5 be the (e_2, w_2) -edge different from the edge e_3 . The edge e_5 is oriented inward at the white vertex w_2 . For, if the edge e_5 is oriented outward at the white vertex w_2 , then by applying a C-I-M2 move between e_1 and e_3 and further applying a C-I-M2 move between e_4 and e_5 , we get three consecutive edges connecting the two white vertices w_1 and w_2 . Hence by applying a C-I-M3 move we can eliminate the two white vertices. This contradicts that the chart is minimal. Therefore the edge e_5 is oriented inward at the white vertex w_2 (see Fig. 19(b)).

Let e_6 be the (e_3, w_2) -edge different from e_2 . Since the three edges e_2, e_3, e_5 are oriented inward at the white vertex w_2 , the edge e_6 is oriented outward at the white vertex w_2 (see Fig. 19(b)).

The vertex w_3 is a white vertex. For, if w_3 is not a white vertex, then w_3 is a crossing or a black vertex. Hence the edge e_3 is not contained in a bigon. By applying a C-I-M2 move between e_1 and e_3 , we can get a new bigon without destroying old bigons. Thus the number of bigons increases. This contradicts that the chart is minimal. Hence the vertex w_3 must be a white vertex.

Since we can apply a C-I-M2 move between e_1 and e_3 , the edge e_3 must contain a middle arc at the white vertex w_3 . Hence the (e_3, w_3) -edges are oriented outward at w_3 . Since the edge e_6 is oriented outward at w_2 (see Fig. 19(c)), neither of the (e_3, w_3) -edges is equal to e_6 . Hence the edge e_3 is not contained in a bigon.

Now by applying a C-I-M2 move between e_1 and e_3 we can get a new bigon without destroying old bigons. Thus the number of bigons increases. This contradicts that the chart is minimal. \square

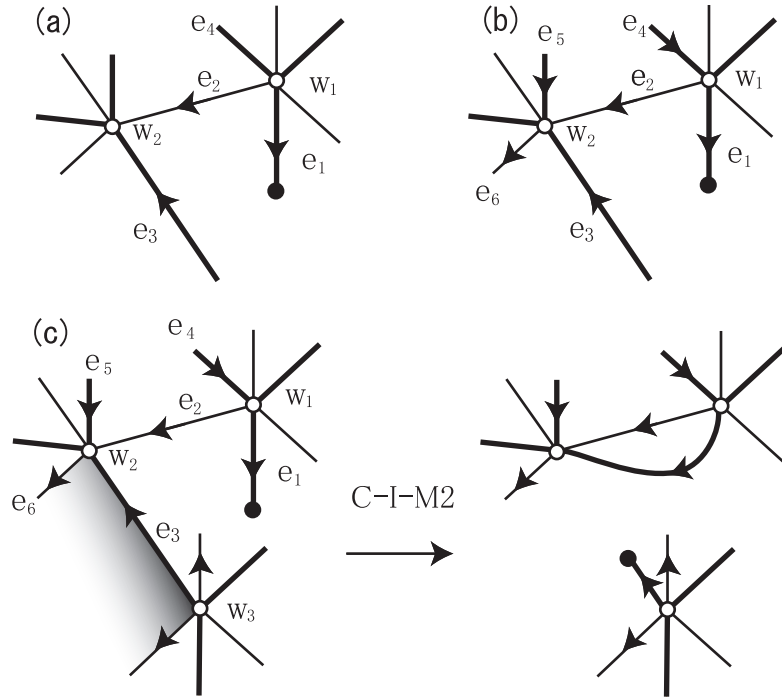


Figure 19:

References

- [1] J. S. Carter and M. Saito, "Knotted surfaces and their diagrams", Mathematical Surveys and Monographs, 55, American Mathematical Society, Providence, RI, (1998). MR1487374 (98m:57027)
- [2] I. Hasegawa, *The lower bound of the w -indices of non-ribbon surface-links*, Osaka J. Math. **41** (2004), 891–909. MR2116344 (2005k:57045)
- [3] S. Ishida, T. Nagase and A. Shima, *Minimal n -charts with four white vertices*, J. Knot Theory Ramifications **20**, 689–711 (2011). MR2806339 (2012e:57044)
- [4] S. Kamada, *Surfaces in R^4 of braid index three are ribbon*, J. Knot Theory Ramifications **1**, No. 2 (1992), 137–160. MR1164113 (93h:57039)
- [5] S. Kamada, *2-dimensional braids and chart descriptions*, Topics in knot theory (Erzurum, 1992), 277–287, NATO Adv. Sci. Inst. Ser. Math. Phys. Sci., 399, Kluwer Acad. Publ., Dordrecht, (1993). MR1257915
- [6] S. Kamada, "Braid and Knot Theory in Dimension Four", Mathematical Surveys and Monographs, Vol. 95, American Mathematical Society, (2002). MR1900979 (2003d:57050)

- [7] T. Nagase and A. Hirota, *The closure of a surface braid represented by a 4-chart with at most one crossing is a ribbon surface*, Osaka J. Math **43** (2006), 413–430. MR2262343 (2007g:57040)
- [8] T. Nagase and A. Shima, *Properties of minimal charts and their applications I*, J. Math. Sci. Univ. Tokyo **14** (2007), 69–97. MR2320385 (2008c:57040)
- [9] T. Nagase and A. Shima, *Properties of minimal charts and their applications II*, Hiroshima Math. J. **39** (2009), 1–35. MR2499196 (2009k:57040)
- [10] T. Nagase and A. Shima, *Properties of minimal charts and their applications III*, Tokyo J. Math. **33** (2010), 373–392. MR2779264 (2012a:57033)
- [11] T. Nagase and A. Shima, *Any chart with at most one crossing is a ribbon chart*, Topology Appl. **157** (2010), 1703–1720. MR2639836 (2011f:57048)
- [12] T. Nagase and A. Shima, *On charts with two crossings I: There exist no NS-tangles in a minimal chart*, J. Math. Sci. Univ. Tokyo **17** (2010), 217–241. MR2759760 (2012a:57032)
- [13] T. Nagase and A. Shima, *On charts with two crossings II*, Osaka J. Math. **49** (2012), 909–929. MR3007949
- [14] T. Nagase and A. Shima, *On charts with three crossings representing 2-knots I, II*, in preparation.
- [15] T. Nagase and A. Shima, *Properties of minimal charts and their applications IV*, in preparation.
- [16] T. Nagase and A. Shima, *The structure of minimal n -charts with two crossings I: Complementary domains of $\Gamma_1 \cup \Gamma_{n-1}$* , preprint.
- [17] T. Nagase and A. Shima, *The structure of minimal n -charts with two crossings II: Neighbourhoods of disks of the lowest label and the highest label*, in preparation.
- [18] M. Ochiai, T. Nagase and A. Shima, *There exists no minimal n -chart with five white vertices*, Proc. Sch. Sci. TOKAI UNIV., 40 (2005), 1–18. MR2138333 (2006b:57035)
- [19] K. Tanaka, *A Note on CI-moves*, Intelligence of Low Dimensional Topology 2006 Eds. J. Scott Carter *et al.* (2006), 307–314. MR2371740 (2009a:57017)

Teruo NAGASE
Tokai University
4-1-1 Kitakaname, Hiratuka
Kanagawa, 259-1292 Japan
nagase@keyaki.cc.u-tokai.ac.jp

Akiko SHIMA
Department of Mathematics,
Tokai University
4-1-1 Kitakaname, Hiratuka
Kanagawa, 259-1292 Japan
shima@keyaki.cc.u-tokai.ac.jp

The List of Notations

Γ_m	$p2$	$\mathcal{I}_M(C)$	$p3, p12$	$P_1^* * P_2^* * \cdots * P_s^*$	$p19$
$\mathcal{O}(C)$	$p3, p12$	$\mathcal{P}(C; \mathcal{S})$	$p25$	$\alpha(X)$	$p41$
$\mathcal{O}_M(C)$	$p3, p12$	$P_1^* * P_2^*$	$p16, p18$	$\beta(X)$	$p41$
$\mathcal{I}(C)$	$p3, p12$				

The List of words

admissible pseudo path	$p19$	middle arc	$p2, p8$
admissible tangle	$p5, p32$	middle at v	$p3, p4$
associated edge sequence	$p17$	monochromatic pseudo path	$p20$
associated side-arc sequence	$p20$	normal edge	$p5$
associated vertex sequence	$p17$	NS-tangle	$p6$
bigon	$p3$	outside edge	$p3$
bridge	$p28$	outward pseudo path	$p5$
co-bridge	$p28$	path	$p4$
consecutive triplet	$p10$	path decomposition $\mathcal{P}(C; \mathcal{S})$	$p25$
co-pier	$p30$	pier	$p29$
cycle of label m	$p3, p12$	point at infinity ∞	$p9$
dichromatic	$p28, p30$	pseudo path	$p4$
edge of Γ_m	$p2$	reducible tree	$p31$
end-arc	$p18$	ring	$p9$
extended pseudo path	$p24$	side-arc	$p4$
finite complementary domain	$p11$	side-disk	$p4$
fundamental information	$p33$	simple hoop	$p10$
hoop	$p7, p9$	small component	$p32$
I/O pair of type I	$p16$	suspicious cycle	$p32$
I/O pair of type II	$p18$	tangle	$p5$
I/O pseudo path	$p5$	tangle induced from G	$p36$
I/O sequence for (P^*, D)	$p19$	tangle induced from a cycle	$p36$
inside edge	$p3$	terminal edge	$p3$
inward pseudo path	$p5$	three-color disk	$p3$
IO-tangle	$p5$	two-color disk	$p24$
loop	$p3$	two-color tangle	$p32$
maximal cycle in X	$p32$		